MATH3401/3901: Complex Analysis/Advanced Complex Analysis

First Semester 2025
16/04/2025

Assignment Number 3

Problem 1 *(6 points)* Show the following limits:
a) $\lim_{z\to\infty} \dfrac{7z^3}{z^3-13z} = 7$; **b)** $\lim_{z\to\infty} \dfrac{z^3}{z^2+13z} = \infty$; **c)** $\lim_{z\to\infty} \dfrac{(az+b)^2}{(cz+d)^2} = \dfrac{a^2}{c^2}$ if $c \neq 0$.

Problem 2 *(3 points)* Show that the following functions are defined on all of $\mathbb{C}$, but are nowhere analytic (here $z = x + iy$):
a) $z \mapsto 2xy + i(x^2 - y^2)$; **b)** $z \mapsto \sin \overline{z}$.

Problem 3 *(4 points)* Show where the function $z \mapsto x^4 + i(1-y)^4$ is:
a) analytic;
b) differentiable
(here $z = x + iy$).

Problem 4 *(3 points)* For $z = x + iy$, define $f(z) = \sqrt{|xy|}$.
a) Show that f satisfies the Cauchy-Riemann equations at the origin. (Note: you will need to use the definition of the partial derivative to calculate $u_x(0,0)$ and $u_y(0,0)$).
b) Show that f is not differentiable at the origin. (Hint: approach on a suitable line).
c) Explain why this doesn't contradict any of the results from class.

Problem 5 *(4 points)*
(a) Define precisely what it means for a curve in $\mathbb{C}$ to be *rectifiable.*
(b) Give an example of a non-rectifiable curve in $\mathbb{C}$. You don't have to prove it is non-rectifiable.
Note: you will need to provide at least one reference, properly cited. This is not allowed to be wikipedia.

Problem 6 *(2 bonus points)* Consider sequences defined as follows:

$$x_0 = 0, \quad x_1 = x_2 = 1, \quad x_3 = x_4 = 1 - \tfrac{1}{2}, \quad x_5 = x_6 = 1 - \tfrac{1}{2} + \tfrac{1}{4}, \cdots \quad \text{and}$$
$$y_0 = y_1 = 0, \quad y_2 = y_3 = 1, \quad y_4 = y_5 = 1 - \tfrac{1}{2}, \quad y_6 = y_7 = 1 - \tfrac{1}{2} + \tfrac{1}{4}, \cdots .$$

Set $z_n = x_n + iy_n$, and define I_n to be the closed line segment (i.e., including end-points) from z_n to z_{n+1}, for $n \in \mathbb{N}_0$. Finally, set:

$$\Omega = \bigcup_{n=0}^{\infty} I_n, \quad \text{and } \Lambda = \Omega \cup \{\tfrac{2}{3} + \tfrac{2}{3}i\}.$$

a) Show that Λ is connected. (A sketch is not a proof, but could be helpful in fixing ideas.)
b) Show that Λ is *not* piecewise affinely path connected, in the sense defined in class (Lecture 12).
c) Do these results contradict anything from class? Explain.

Due: 10:00AM, Friday, 02/05/2025

Current assignments will be available at
`http://www.maths.uq.edu.au/courses/MATH3401/AssignmentsEtc.html`