MATH3901: Advanced Complex Analysis

First Semester 2026
01/05/2026

Assignment Number 4

Problem 1 *(2 points)* Define

$$f(z) = \begin{cases} z^5/|z^4| & z \neq 0 \\ 0 & z = 0. \end{cases}$$

(a) Show that f satisfies the Cauchy-Riemann equations at the origin. (Note: you will need to use the definition of the partial derivative to calculate the partial derivatives of u and v at the origin.)
(b) Show that f is not differentiable at the origin.
(c) Explain why this doesn't contradict any of the results from class.

Problem 2 *(2 points)*
Verify that the following functions u are harmonic, and in each case give a conjugate harmonic function v (i.e., v such that $u + iv$ is analytic).
(a) $e^x \cos y$,
(b) $x^2 - y^2 - 2y$.

Problem 3 *(4 points)*
(a) Suppose that U solves a Neumann problem for Laplace's equation on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Show that $U + c$ also solves this problem for any $c \in \mathbb{R}$.
(b) Does the same result hold for the corresponding Dirichlet problem?

Problem 4 *(2 points)*
Find a power-series expansion of the function $f(z) = \dfrac{1}{3-z}$ about the point $4i$, and calculate its radius of convergence.

Problem 5 *(2 points)*
Find a Laurent-series expansion of the function $f(z) = z^{-1}\sinh(z^{-1})$ about the point 0, and classify the singularity at 0.

Problem 6 *(2 points)*
Show that, for all $r \in [0, 1)$, there holds:

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}.$$

Problem 7 *(3 points)*
(a) Prove that the coefficients c_n in the expansion

$$\frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} c_n z^n$$

satisfy the recurrence relation $c_0 = c_1 = 1$, $c_n = c_{n-1} + c_{n-2}$ for $n \geq 2$.
(b) What is the radius of convergence of the series in (a)?
(c) What would be a good name for the c_n's?

(...OVER)

Problem 8 *(3 points)*
Show that, for all $n \in \mathbb{N}$, there holds:

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} 2\pi \, .$$

Hint: Put $z = e^{i\theta}$ and rewrite the integral.

Due: 10:00AM, Friday, 22/05/2026

Current assignments will be available at
`http://www.maths.uq.edu.au/courses/MATH3401/AssignmentsEtc.html`