MATH3401/3901 Problem Worksheet Semester 1, 2025, Week 10

(1) Determine where the function $f(z) = z - e^{-z} + 1 - i$ is conformal.

Solution: It is easy to verify that $f(z) = z - e^{-z} + 1 - i$ is entire on \mathbb{C} . Now $f'(z) = 1 + e^{-z} = 0$ if

$$e^{z}(1 + e^{-z}) = e^{z}(0)$$

$$e^{z} + 1 = 0$$

$$e^{z} = -1$$

$$z = \log(-1) = \ln|-1| + i \arg(-1)$$

$$z = i(\pi + 2\pi n)$$

$$z = (2n + 1)\pi i,$$

for $n \in \mathbb{Z}$. Hence, f is conformal for all z except $z = (2n+1)\pi i, n \in \mathbb{Z}$.

(2) Find all points where the function $f(z) = \sin z$ is conformal.

Solution: The function $f(z) = \sin z$ is entire on \mathbb{C} and we have that $f'(z) = \cos z$. Now $\cos z = 0$ if and only if $z = (2n + 1)\pi/2$, $n \in \mathbb{Z}$ and so each of these points is a critical point of f. Therefore, f is conformal for all z except $z = (2n + 1)\pi/2$, $n \in \mathbb{Z}$.

(3) Show that the angle of rotation at a nonzero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n$ (n = 1, 2, ...) is $(n - 1)\theta_0$. Determine the scale factor of the transformation at that point.

Solution: Notice that $f'(z) = nz^{n-1}$. Now for $z_0 = r_0 e^{i\theta_0}$ we have

$$f'\left(r_0e^{i\theta_0}\right) = n'\left(r_0e^{i\theta_0}\right)^{n-1}.$$

Thus the angle of rotation is

$$\arg\left[f'\left(r_0e^{i\theta_0}\right)\right] = \arg\left[n\left(r_0e^{i\theta_0}\right)^{n-1}\right] = (n-1)\theta_0$$

and the scale factor is

$$\left| f'(r_0 e^{i\theta_0}) \right| = \left| n r_0^{n-1} e^{i(n-1)\theta_0} \right| = n r_0^{n-1}.$$

See Section 112 in the Brown and Churchill textbook for further discussion on the angle of rotation.

(4) Find a function harmonic in the upper half of the z-plane, Im z > 0, which takes the prescribe values on the x axis given by G(x) = 1 for x > 0, and G(x) = 0 for x < 0.

Solution: We need to find $\Phi(x, y)$ such that

$$abla^2 \Phi = 0, \quad -\infty < x < \infty, \quad y > 0.$$

 $\Phi = 1, \quad x > 0; \quad \Phi = 0, \quad x < 0.$

This is a Dirichlet problem for the upper half plane.



Figure 1: Upper half plane.

The function $A\theta + B$, where A and B are real constants, is harmonic since it is the imaginary part of $A \log(z) + Bi$. To determine A and B note that the boundary conditions are $\Phi = 1$ for x > 0, that is, $\theta = 0$ and $\Phi = 0$ for x < 0, that is $\theta = \pi$. Thus

$$1 = A(0) + B, \qquad 0 = A(\pi) + B$$

from which $A = -1/\pi$, B = 1. Then the required solution is

$$\Phi = A\theta + B = 1 - \frac{\theta}{\pi} = 1 - \frac{1}{\pi} \operatorname{Arg} z$$

We can write this as

$$\Phi(x,y) = 1 - \frac{1}{\pi}\arctan(y/x)$$

for y > 0, where we take arctan to be the inverse of tan taking values in $(0, \pi]$. Notice for this arctan,

$$\lim_{x \to 0^+} \arctan(x) = 0, \qquad \lim_{x \to 0^-} \arctan(x) = \pi$$

(indeed, draw a graph of $y = \tan(x)$ with x restricted to $(0, \pi]$). It follows that

$$\lim_{y \to 0^+} \Phi(x, y) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

(5) Find a function harmonic inside the unit circle |z| = 1 and taking the prescribed values given by $F(\theta) = 1$ for $0 < \theta < \pi$, and $F(\theta) = 0$ for $\pi < \theta < 2\pi$ on its circumference.

Solution: This is a Dirichlet problem for the unit circle in which we need to find a function satisfying Laplace's equation inside |z| = 1 and taking the values 1 on the upper arc of the circle and 0 on the lower arc of the circle.



Figure 2: Unit circle.

Figure 3: Image under w = i(1-z)/(1+z).

We map the interior of the circle |z| = 1 on to the upper half of the w plane by using the mapping

$$w = i \frac{1-z}{1+z}.$$

Under this transformation, the upper and lower arcs are mapped on to the positive and negative real axis on the *w*-plane respectively. This means that the boundary conditions $\Phi = 1$ on the upper arc of the circle and $\Phi = 0$ on the lower arc of the circle become respectively $\Phi = 1$ for u > 0 and $\Phi = 0$ for u < 0.

Thus we have reduced the problem to finding a function Φ harmonic in the upper half *w*-plane and taking the values 0 for u < 0 and 1 for u > 0. But this problem has already been solved in the previous exercise, and the solution (replacing x by u and y by v) is given by

$$\Phi = 1 - \frac{1}{\pi} \arctan\left(\frac{v}{u}\right). \tag{1}$$

Now from w = i(1-z)/(z+1), we find

$$u = \frac{2y}{(1+x)^2 + y^2}, \quad v = \frac{1 - (x^2 + y^2)}{(1+x)^2 + y^2}.$$

Then substituting these in (1), we find the required solution

$$\Phi = 1 - \frac{1}{\pi} \arctan\left(\frac{1 - (x^2 + y^2)}{2y}\right)$$

or in polar coordinates (r, θ)

$$\Phi = 1 - \frac{1}{\pi} \arctan\left(\frac{1 - r^2}{2r\sin\theta}\right).$$