SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Problem Worksheet
Semester 1, 2025, Week 11

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(1) Use the definition of limits of sequences, given in Lecture 30 page 3, to verify that the limit of the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots).$$

converges to -2.

Solution: Notice that

$$\big|z_n - (-2)\big| = \frac{1}{n^2}.$$

Thus, for each $\varepsilon > 0$,

$$\big|z_n - (-2)\big| < \varepsilon \quad \text{whenever} \quad n > n_0,$$

where n_0 is any positive integer such that $n_0 \geq \dfrac{1}{\sqrt{\varepsilon}}$.

(2) Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4 + 9}$$

and calculate the radius of convergence.

Solution: We want the Maclaurin series for

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{1 - (-z^4/9)}$$

Replace just z by $(-z^4)/9$ in

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

as well as its condition of validity, to get

$$\frac{1}{1 + (-z^4/9)} = \sum_{n=0}^{\infty} \left(\frac{-z^4}{9} \right)^n = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} z^{4n}}_{\text{Final result}} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n}}_{\text{Extra simplification}}, \quad |z| < \sqrt{3}.$$

Then if we multiply through this last equation by $z/9 = z/3^2$, we have the desired expansion:

$$f(z) = \frac{z}{3^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1}, \quad |z| < \sqrt{3}.$$

(3) Find the Taylor series of the function

$$f(z) = \frac{1}{1-z}$$

about the point i and provide the radius of convergence.

Solution: The function $\dfrac{1}{1-z}$ has a singularity at $z = 1$. So the Taylor series about $z = i$ is valid when $\left|z - i\right| < \sqrt{2}$.

To find the series, we start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}.$$

Now we can replace z by $(z-i)/(1-i)$ in the known expression

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (\left|z\right| < 1)$$

and then multiply through by $\dfrac{1}{1-i}$. Therefore, the Taylor series is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (\left|z-i\right| < \sqrt{2}).$$

(4) Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Solution: Suppose that $0 < |z| < 4$. Then $0 < |z/4| < 1$, and we can use the know expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

That is, when $0 < |z| < 4$

$$\begin{aligned}
\frac{1}{4z - z^2} &= \frac{1}{4z} \cdot \frac{1}{1 - \frac{z}{4}} = \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}} \\
&= \frac{1}{4z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}} \\
&= \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}
\end{aligned}$$

(5) Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Hint 1: For one domain you should get

$$\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z}.$$

For the other domain, you should get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}.$$

Hint 2: Observe that $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$.

Solution:

The function $f(z)$ has isolated singularities at $z = 0$ and $z = \pm i$.

Hence there is a Laurent series representation for the domain $0 < |z| < 1$ and also one for the domain $1 < |z| < \infty$, which is exterior to the circle $|z| = 1$.

To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

For the domain $0 < |z| < 1$, we have

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1+z^2} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-z^2\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n z^{2n-1} \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z}. \end{aligned}$$

On the other hand, when $1 < |z| < \infty$,

$$
\begin{aligned}
f(z) &= \frac{1}{z^3}\frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}
\end{aligned}
$$

In this last part we use the fact that $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$.