SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Problem Worksheet
Semester 1, 2025, Week 12

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(1) Use Cauchy's residue theorem (Lecture 33) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

$$(a)\ \frac{\exp(-z)}{z^2}; \qquad (b)\ z^2 \exp\left(\frac{1}{z}\right); \qquad (c)\ \frac{z+1}{z^2-2z}.$$

Solution: In each part, C denotes the positively oriented circle $|z| = 3$.

(a) We need to evaluate

$$\int_C \frac{\exp(-z)}{z^2}\, dz.$$

So we have to calculate the residue of the integrand at $z = 0$. From the Laurent series

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right) = \frac{1}{z^2} - \frac{1}{1!} \cdot \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots \qquad (0 < |z| < \infty),$$

and we see that the required residue is -1. Therefore

$$\int_C \frac{\exp(-z)}{z^2}\, dz = 2\pi i(-1) = -2\pi i.$$

(b) Now we need to evaluate

$$\int_C z^2 \exp\left(\frac{1}{z}\right)\, dz.$$

To find the residue at $z = 0$ consider the Laurent series

$$\begin{aligned} z^2 \exp\left(\frac{1}{z}\right) &= z^2\left(1 + \frac{1}{1!} \cdot \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} \cdots\right) \\ &= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdot \frac{1}{z} + \frac{1}{4!} \cdot \frac{1}{z^2} + \cdots \end{aligned}$$

which is valid for $0 < |z| < \infty$. Then the residue is $\frac{1}{6}$. Hence

$$\int_C z^2 \exp\left(\frac{1}{z}\right) dz = 2\pi i \left(\frac{1}{6}\right) = \frac{\pi i}{3}.$$

(c) Finally, to evaluate

$$\int_C \frac{z+1}{z^2-2z}\, dz.$$

we need the two residues of

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)},$$

one at $z = 0$ and one at $z = 2$. The residue at $z = 0$ can be found by writing

$$\begin{aligned}\frac{z+1}{z(z-2)} &= \left(\frac{z+1}{z}\right)\left(\frac{1}{z-2}\right) = \left(-\frac{1}{2}\right)\left(1+\frac{1}{z}\right)\cdot\frac{1}{1-(z/2)} \\ &= \left(-\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{z}\right)\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right),\end{aligned}$$

which is valid when $0 < |z| < 2$, and observing that the coefficient of $\frac{1}{z}$ in this last product is $-\frac{1}{2}$. To obtain the residue at $z = 2$, we write

$$\begin{aligned}\frac{z+1}{z(z-2)} &= \frac{(z-2)+3}{z-2}\cdot\frac{1}{2+(z-2)} = \frac{1}{2}\left(1+\frac{3}{z-2}\right)\cdot\frac{1}{1+(z-2)/2} \\ &= \frac{1}{2}\left(1+\frac{3}{z-2}\right)\left[1-\frac{z-2}{2}+\frac{(z-2)^2}{2^2}-\cdots\right],\end{aligned}$$

which is valid when $0 < |z-2| < 2$, and note that the coefficient of $\frac{1}{z-2}$ in this product is $\frac{3}{2}$. Therefore, by the residue theorem,

$$\int_C \frac{z+1}{z^2-2z}\, dz = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = 2\pi i.$$

(2) In each case, find the Laurent series of the function at its isolated singular point. Determine whether that point is a pole (determine its order), a removable singular point or an essential singularity. Finally, determine the corresponding residue.

$$(a)\ z\exp\left(\frac{1}{z}\right); \quad (b)\ \frac{z^2}{1+z}; \quad (c)\ \frac{\cos z}{z}; \quad (d)\ \frac{1-\cosh z}{z^3}; \quad (e)\ \frac{1}{(2-z)^3}.$$

Suggestion 1: Use the known series

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}, \qquad \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}, \qquad \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}, \qquad (|z|<\infty).$$

Suggestion 2: For part (b) notice that $z^2 = (z+1)^2 - 2z - 1 = (z+1)^2 - 2(z+1) + 1$

Solution:

(a) From the expansion

$$\exp(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

we see that

$$f(z) = z\exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^3} + \cdots$$

The principal part of $z\exp\left(\frac{1}{z}\right)$ at the isolated singular point $z=0$ is then

$$\frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^3} + \cdots$$

and $z=0$ is an essential singularity. Finally,

$$b_1 = \operatorname*{\mathbf{Res}}_{z=0} f(z) = \frac{1}{2!}.$$

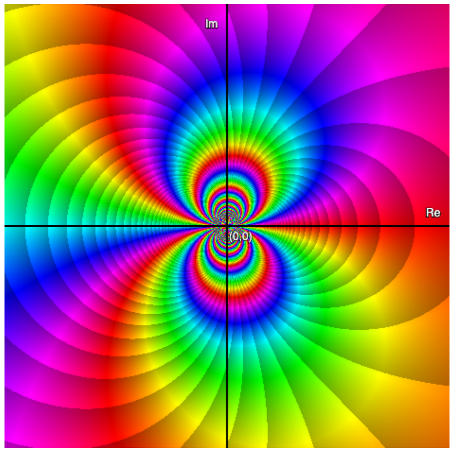

Figure 1: Domain coloring for $z \exp(1/z)$. Link: Domain Coloring

(b) The isolated singular point of

$$f(z) = \frac{z^2}{1+z}$$

is at $z = -1$. Using suggestion 2 we have that

$$\frac{z^2}{1+z} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1} = (z+1) - 2 + \frac{1}{z+1}$$

In this case the principal part is $\dfrac{1}{z+1}$, and the point $z = -1$ is a simple pole. Finally,

$$b_1 = \underset{z=-1}{\mathbf{Res}}\, f(z) = 1.$$

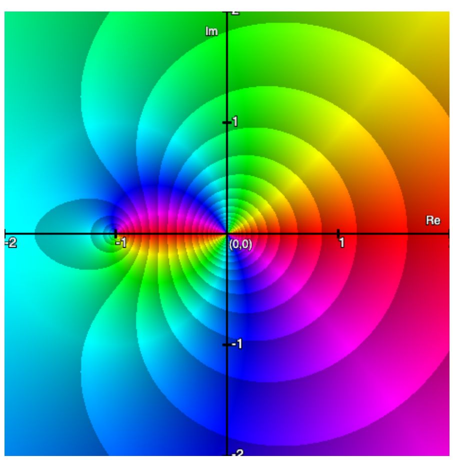

Figure 2: Domain coloring for $z^2/(1+z)$. Link: Domain Coloring

(c) The isolated singular point of

$$f(z) = \frac{\cos z}{z}$$

is $z = 0$. Using the known series

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

we have

$$\frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots$$

Thus the principal part is $\frac{1}{z}$. This means that $z = 0$ is a simple pole. Finally

$$b_1 = \mathop{\mathbf{Res}}_{z=0} f(z) = 1.$$

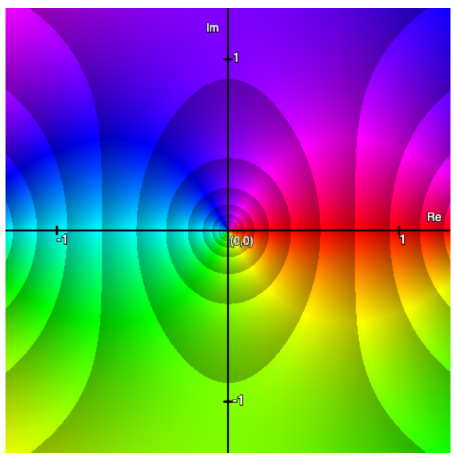

Figure 3: Domain coloring for $\cos z/z$. Link: Domain Coloring

(d) The singular point in this case is $z = 0$. Using the known series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

we have

$$\begin{aligned} \frac{1 - \cosh z}{z^3} &= \frac{1}{z^3}\left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right)\right] \\ &= -\frac{1}{2!} \cdot \frac{1}{z} - \frac{z}{4!} - \frac{z^3}{6!} - \cdots \end{aligned}$$

Thus the principal part is $-\frac{1}{2} \cdot \frac{1}{z}$. This means that $z = 0$ is a simple pole. Finally

$$b_1 = \operatorname*{\mathbf{Res}}_{z=0} f(z) = -\frac{1}{2!}.$$

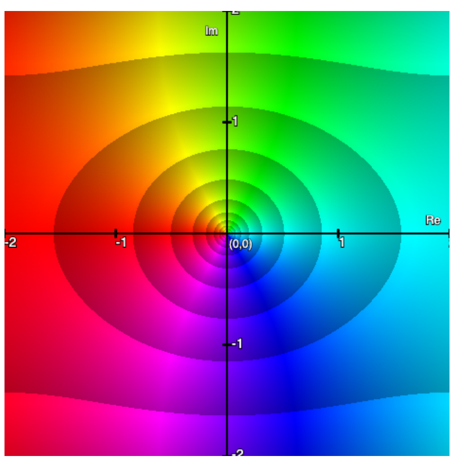

Figure 4: Domain coloring for $(1 - \cosh z)/z^3$. Link: Domain Coloring

(e) The function $f(z) = \dfrac{1}{(2-z)^3}$ has a singular point at $z = 2$. Notice also that

$$\frac{1}{(2-z)^3} = \frac{-1}{(z-2)^3}.$$

In this case the principal part of f is the function itself. The singular point is a pole of order 3 and

$$b_1 = \underset{z=2}{\mathbf{Res}}\, f(z) = 0.$$

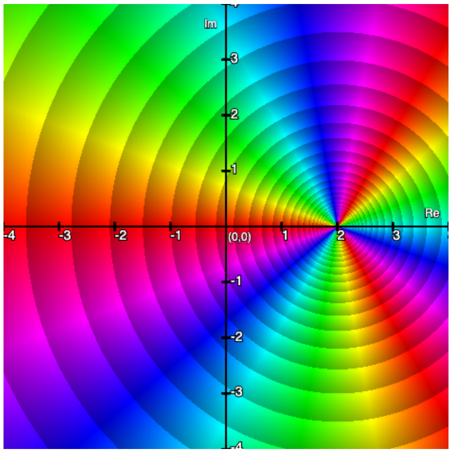

Figure 5: Domain coloring for $1/(2-z)^3$. Link: Domain Coloring

(3) Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

taken counterclockwise around the circle (a) $|z-2|=2$; (b) $|z|=4$.

Ans. (a) πi; *(b)* $6\pi i$.

Solution - Part (a):

Observe that the point $z_0 = 1$, which is the only singularity inside C, is a simple pole of the integrand.

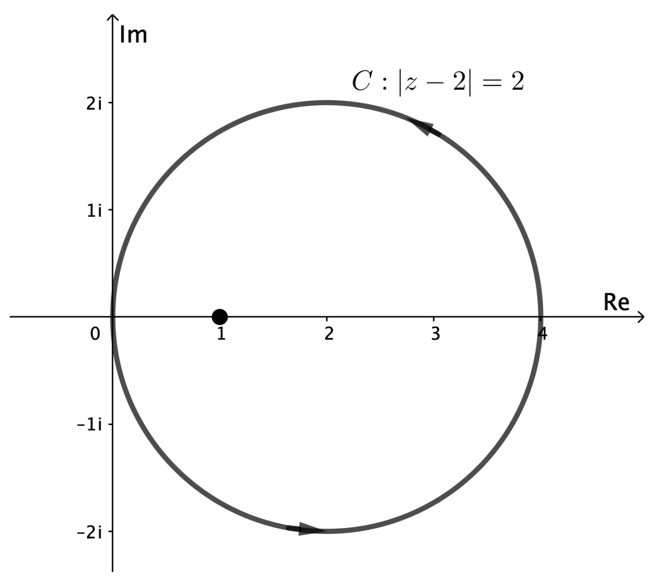

Figure 6: Circle $|z-2|=2$.

Notice that

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z-1} \quad \text{with} \quad \phi(z) = \frac{3z^3+2}{z^2+9}.$$

Since $\phi(z)$ is analytic at $z_0 = 1$ and $\phi(z_0) \neq 0$, then

$$\underset{z=1}{\mathbf{Res}} \frac{3z^3+2}{(z-1)(z^2+9)} = \left.\frac{3z^3+2}{z^2+9}\right|_{z=1} = \frac{3(1)^3+2}{(1)^2+9} = \frac{5}{10} = \frac{1}{2}.$$

Hence

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i \cdot \frac{1}{2} = \pi i.$$

Solution - Part (b):

In this case the singularities $z_0 = 1, z_1 = 3i, z_2 = -3i$ of the integrand are inside C.

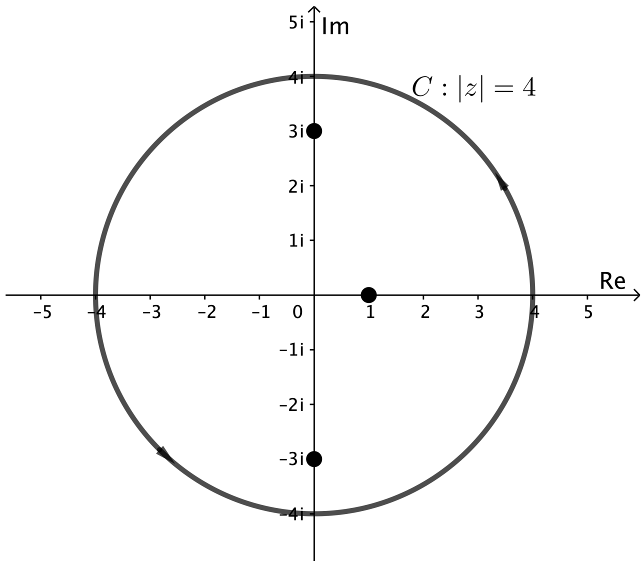

Figure 7: Circle $|z| = 4$.

From part (a)

$$\underset{z=1}{\text{Re}} \frac{3z^3 + 2}{(z-1)(z^2+9)} = \frac{1}{2}.$$

Now, notice that

$$\frac{3z^3 + 2}{(z-1)(z^2+9)} = \frac{3z^3 + 2}{(z-1)(z+3i)(z-3i)}$$

Thus for $z_1 = 3i$ we have

$$\frac{3z^3 + 2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z - 3i} \quad \text{with} \quad \phi(z) = \frac{3z^3 + 2}{(z-1)(z+3i)}.$$

Since $\phi(z)$ is analytic at $z_1 = 3i$ and $\phi(z_1) \neq 0$, then

$$\underset{z=3i}{\mathbf{Res}} \frac{3z^3 + 2}{(z-1)(z^2+9)} = \left. \frac{3z^3 + 2}{(z-1)(z+3i)} \right|_{z=3i} = \frac{3(3i)^3 + 2}{((3i) - 1)((3i) + 3i)} = \frac{15 + 49i}{12}.$$

On the other hand, for $z_2 = -3i$ we have

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z+3i} \quad \text{with} \quad \phi(z) = \frac{3z^3+2}{(z-1)(z-3i)}.$$

Since $\phi(z)$ is analytic at $z_2 = -3i$ and $\phi(z_2) \neq 0$, then

$$\mathop{\mathbf{Res}}_{z=-3i} \frac{3z^3+2}{(z-1)(z^2+9)} = \left.\frac{3z^3+2}{(z-1)(z-3i)}\right|_{z=-3i} = \frac{3(-3i)^3+2}{((-3i)-1)((-3i)-3i)} = \frac{15-49i}{12}.$$

Therefore, using Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathop{\mathbf{Res}}_{z=z_k} f(z),$$

we find that

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i\left(\frac{1}{2} + \frac{15+49i}{12} + \frac{15-49i}{12}\right) = 2\pi i(3) = 6\pi i.$$