SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401
Problem Worksheet
Semester 1, 2025, Week 13

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(1) Find the Laurent series expansion of

$$f(z) = (1 - z^2)\sin\left(\frac{1}{z}\right)$$

about the point $z = 0$, classify the singularity, and find the residue at that point.

Solution: The function has a singularity at 0. Its Laurent series is:

$$\begin{aligned}
(1 - z^2)\sin\left(\frac{1}{z}\right) &= (1 - z^2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n-1}} \\
&= -z + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}} - \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n-1}} \\
&= -z + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+3)!z^{2n+1}} \\
&= -z + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+3)!z^{2n+1}} \\
&= -z + \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(2n+1)!} + \frac{1}{(2n+3)!}\right)\frac{1}{z^{2n+1}}.
\end{aligned}$$

Therefore, the function has an essential singularity at 0 and

$$\operatorname*{\mathbf{Res}}_{z=0} f(z) = \frac{1}{1!} + \frac{1}{3!} = \frac{7}{6}.$$

(2) Use residues to evaluate the improper integral:

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

Ans. $\pi/4$.

Solution: First notice that the function $1/(x^2+1)^2$ is even. Then

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}.$$

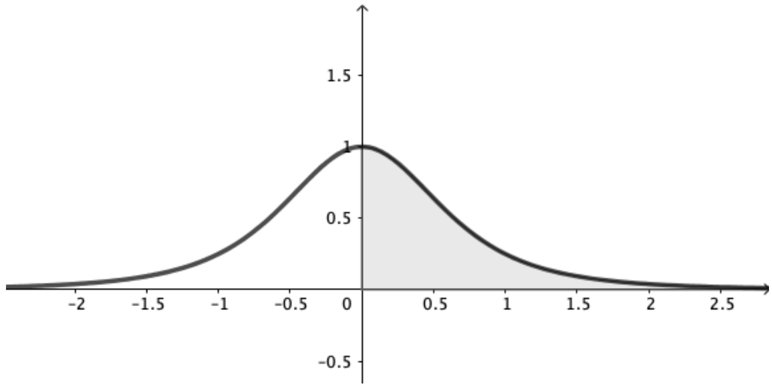

Figure 1: Improper integral.

Now we need to calculate the integral

$$\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}.$$

To do this we will calculate the integral of the complex function

$$f(z) = \frac{1}{(z^2+1)^2}$$

around the simple closed contour consisting of:

(i) the segment of the real axis from $z = -R$ to $z = R$, and

(ii) the top half of the circle $|z| = R$, described counterclockwise and denoted by C_R

with $R > 1$, see Figure **??**.

Since the singularity $z_0 = i$ lies in the interior of $C_R\,(R > 1)$, we have that

$$\int_{-R}^R \frac{dx}{(x^2+1)^2} + \int_{C_R} \frac{dz}{(z^2+1)^2} = 2\pi i B,$$

where

$$B = \operatorname*{\mathbf{Res}}_{z=i} \frac{1}{(z^2+1)^2}.$$

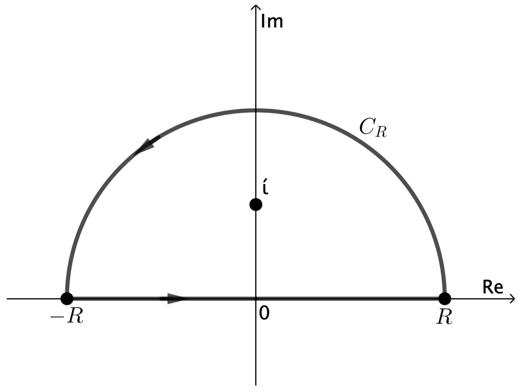

Figure 2: Simple closed contour.

Since

$$\frac{1}{(z^2+1)^2} = \frac{\phi(z)}{(z-i)^2}, \quad \text{where} \quad \phi(z) = \frac{1}{(z+i)^2},$$

we can find that $B = \phi^{(1)}(i) = \dfrac{1}{4i}$ (Why?). Thus

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2}.$$

Observe that if $z \in C_R$,

$$|z^2+1| \geq ||z|^2 - 1| = R^2 - 1.$$

Thus

$$\left| \int_{C_R} \frac{dz}{(z^2+1)^2} \right| \leq \frac{\pi R}{(R^2-1)^2} = \frac{\frac{\pi}{R^3}}{\left(1 - \frac{1}{R^2}\right)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

Then

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}.$$

Therefore

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} = \frac{\pi}{4}.$$

(3) Use residues to find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}.$$

Ans. $-\pi/5$.

Solution: Here we need to show that

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = -\frac{\pi}{5}.$$

To do this we introduce the function

$$f(z) = \frac{z}{(z^2+1)(z^2+2z+2)}$$

and the simple closed contour shown below.

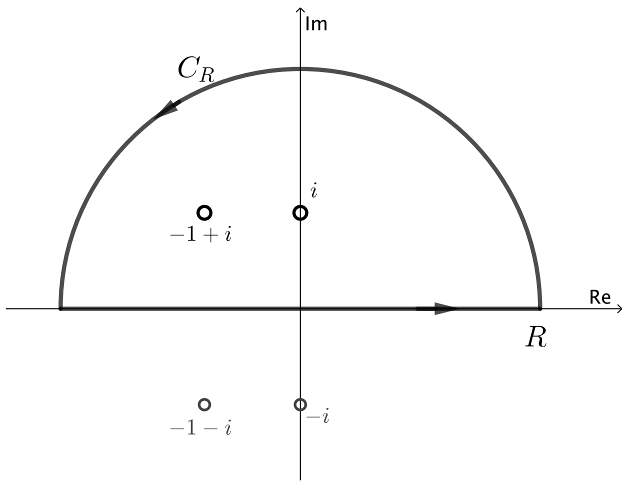

Figure 3: Simple closed contour.

Notice that the singularities of $f(z)$ are at i, $z_0 = -1+i$ and their conjugates $-i$, $\overline{z_0} = -1-i$ in the lower half plane. Also, if $R > \sqrt{2}$, we see that

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i(B_0 + B_1),$$

where

$$B_0 = \operatorname*{\mathbf{Res}}_{z=z_0} f(z) = \left[\frac{z}{(z^2+1)(z-\overline{z_0})}\right]_{z=z_0} = -\frac{1}{10} + \frac{3}{10}i$$

and

$$B_1 = \operatorname*{\mathbf{Res}}_{z=i} f(z) = \left[\frac{z}{(z+i)(z^2+2z+2)}\right]_{z=i} = \frac{1}{10} - \frac{1}{5}i.$$

Thus

$$\int_{-R}^{R} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}\,dx = -\frac{\pi}{5} - \int_{C_R} \frac{z\,dz}{(z^2+1)(z^2+2z+2)}.$$

Since

$$\left|\int_{C_R} \frac{z\,dz}{(z^2+1)(z^2+2z+2)}\right| = \left|\int_{C_R} \frac{z\,dz}{(z^2+1)(z-z_0)(z-\overline{z_0})}\right| \leq \frac{\pi R^2}{(R^2-1)(R-\sqrt{2})^2}$$

converges to 0 as R goes to infinity, then

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = -\frac{\pi}{5}.$$

which is the required result.

(4) Determine the number of zeros, counting multiplicities, of the polynomials

(a) $z^4 + 3z^3 + 6$;

(b) $z^4 - 2z^3 + 9z^2 + z - 1$;

(c) $z^5 + 3z^3 + z^2 + 1$

inside the circle $|z| = 2$.

Ans. (a) 3; (b) 2; (c) 5.

Solution: Let C denote the circle $|z| = 2$.

(a) The polynomial $z^4 + 3z^3 + 6$ can be written as the sum of the polynomials

$$f(z) = 3z^3 \quad \text{and} \quad g(z) = z^4 + 6.$$

On C,

$$|f(z) = 3|z|^3 = 24 \quad \text{and} \quad |g(z)| = |z^4 + 6| \leq |z|^4 + 6 = 22.$$

Since $|f(z)| > |g(z)|$ on C and $f(z)$ has 3 zeros, counting multiplicities, inside C, it follows that the original polynomial has 3 zeros, counting multiplicities, inside C.

(b) The polynomial $z^4 - 2z^3 + 9z^2 + z - 1$ can be written as the sum of the polynomials

$$f(z) = 9z^2 \quad \text{and} \quad g(z) = z^4 - 2z^3 + z - 1.$$

On C,

$$|f(z) = 9|z|^2 = 36 \quad \text{and} \quad |g(z)| = |z^4 - 2z^3 - 1| \leq |z|^4 + 2|z|^3 + |z| + 1 = 35.$$

Since $|f(z)| > |g(z)|$ on C and $f(z)$ has 2 zeros, counting multiplicities, inside C, it follows that the original polynomial has 2 zeros, counting multiplicities, inside C.

(c) The polynomial $z^5 + 3z^3 + z^2 + 1$ can be written as the sum of the polynomials

$$f(z) = z^5 \quad \text{and} \quad g(z) = 3z^3 + z^2 + 1.$$

On C,

$$|f(z) = |z|^5 = 32 \quad \text{and} \quad |g(z)| = |3z^3 + z^2 + 1| \leq 3|z|^3 + |z|^2 + 1 = 29.$$

Since $|f(z)| > |g(z)|$ on C and $f(z)$ has 5 zeros, counting multiplicities, inside C, it follows that the original polynomial has 5 zeros, counting multiplicities, inside C.