

SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401/3901

Problem Worksheet

Semester 1, 2025, Week 2

(1) Prove that multiplication of complex numbers is commutative.

Solution. Consider $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ complex numbers. Now

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) \\ &= (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2) \\ &= (x_2 x_1 - y_2 y_1, y_2 x_1 + x_2 y_1) \\ &= (x_2, y_2)(x_1, y_1) \\ &= z_2 z_1. \end{aligned}$$

This proves that multiplication of complex numbers is commutative.

(2) Simplify each of these to a real number:

$$(a) \frac{1+2i}{3-4i} + \frac{2-i}{5i}; \quad (b) \frac{5i}{(1-i)(2-i)(3-i)}.$$

Solution.

$$\begin{aligned} (a) \frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)(-5i)}{(5i)(-5i)} \\ &= \frac{-5+10i}{25} + \frac{-5-10i}{25} = -\frac{2}{5} \end{aligned}$$

$$\begin{aligned} (b) \frac{5i}{(1-i)(2-i)(3-i)} &= \frac{5i}{(1-3i)(3-i)} \\ &= \frac{5i}{-10i} = -\frac{1}{2} \end{aligned}$$

(3) Use the properties of conjugates and moduli to show that

$$(a) \overline{\bar{z} + 3i} = z - 3i; \quad (b) \overline{iz} = -i\bar{z}; \quad (c) \left| (2\bar{z} + 5)(\sqrt{2} - i) \right| = \sqrt{3}|2z + 5|.$$

Solution.

$$\begin{aligned} (a) \overline{\bar{z} + 3i} &= \bar{\bar{z}} + \overline{3i} \\ &= z - 3i. \end{aligned}$$

$$(b) \overline{iz} = \bar{i} \bar{z} = -i\bar{z}.$$

$$\begin{aligned} (c) \left| (2\bar{z} + 5)(\sqrt{2} - i) \right| &= |2\bar{z} + 5| \left| \sqrt{2} - i \right| \\ &= |2\bar{z} + 5| \sqrt{2 + 1} \\ &= \sqrt{3}|2z + 5|. \end{aligned}$$

(4) Verify that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Solution. To verify the inequality $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$, write $z = x + iy$, with $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Hence, we need to show

$$\begin{aligned} \sqrt{2}\sqrt{x^2 + y^2} &\geq |x| + |y|, \text{ or on squaring both sides} \\ 2(x^2 + y^2) &\geq |x|^2 + 2|x||y| + |y|^2. \end{aligned}$$

Since all quantities are non-negative, these expressions are equivalent. On subtracting the RHS from both sides, this becomes equivalent to

$$\begin{aligned} |x|^2 - 2|x||y| + |y|^2 &\geq 0, \text{ i.e.,} \\ (|x| - |y|)^2 &\geq 0. \end{aligned}$$

This last inequality is true (square of a real number), hence so is the original inequality.

(5) Use de Moivre's formula to derive the following trigonometric identities:

$$(a) \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta; \quad (b) \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

Solution. We know from de Moivre's formula that

$$(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$$

or

$$\cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$$

In other words,

$$(\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) = \cos 3\theta + i \sin 3\theta$$

By equating the corresponding real and imaginary parts, we arrive at the desired trigonometric identities.