

SCHOOL OF MATHEMATICS AND PHYSICS

MATH3401

Problem Worksheet

Semester 1, 2025, Week 7

(1) Evaluate the following integrals:

a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

b) $\int_0^{\pi/6} e^{i2t} dt$

Solution.

$$\begin{aligned}\int_1^2 \left(\frac{1}{t} - i\right)^2 dt &= \int_1^2 \left(\frac{1}{t^2} - 1\right) dt - 2i \int_1^2 \frac{1}{t} dt \\ &= -\frac{1}{2} - 2i \ln(2) = -\frac{1}{2} - i \ln(4)\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/6} e^{i2t} dt &= \left[\frac{e^{i2t}}{2i} \right]_0^{\pi/6} \\ &= \frac{1}{2i} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} - 1 \right) \\ &= \frac{\sqrt{3}}{4} + \frac{i}{4}\end{aligned}$$

(2) Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0, & \text{when } m \neq n, \\ 2\pi, & \text{when } m = n. \end{cases}$$

Solution. First, notice that

$$I = \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta.$$

If $m = n$, I becomes

$$I = \int_0^{2\pi} d\theta = 2\pi.$$

If $m \neq n$, then

$$I = \left[\frac{e^{i(m-n)\theta}}{i(m-n)} \right]_0^{2\pi} = \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0.$$

(3) Evaluate $\int_C f(z)dz$ for $f(z) = (z + 2)/z$ and C is

- a) the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$);
- b) the semicircle $z = 2e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$);
- c) the circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq 2\pi$).

Solution. For part a) we have

$$\begin{aligned}\int_C \frac{z+2}{z} dz &= \int_C \left(1 + \frac{2}{z}\right) dz = \int_0^\pi \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta} d\theta \\&= 2i \int_0^\pi (e^{i\theta} + 1) d\theta \\&= 2i \left[\frac{e^{i\theta}}{i} + \theta \right]_0^\pi = 2i(i + \pi + i) \\&= -4 + 2\pi i\end{aligned}$$

For part b) we have

$$\begin{aligned}\int_C \frac{z+2}{z} dz &= 2i \int_\pi^{2\pi} (e^{i\theta} + 1) d\theta \\&= \left[\frac{e^{i\theta}}{i} + \theta \right]_\pi^{2\pi} = 2i(-i + 2\pi - i - \pi) \\&= 4 + 2\pi i\end{aligned}$$

For part c) we just add the previous results to obtain $4\pi i$.

(4) Find the contour integral $\int_C \bar{z} dz$ for

- (a) C is the triangle XYZ oriented counterclockwise, where $X = 0$, $Y = 1 + i$ and $Z = -2$;
- (b) C is the circle $|z - i| = 2$ oriented counterclockwise.

Solution. For part (a) we have

$$\begin{aligned}
 \int_C \bar{z} dz &= \int_{XY} \bar{z} dz + \int_{YZ} \bar{z} dz + \int_{ZX} \bar{z} dz \\
 &= \int_0^1 \underbrace{t(1+i)}_{f(z(t))} \cdot \underbrace{(1+i)}_{z'(t)} dt \\
 &\quad + \int_0^1 \underbrace{(1-t)(1+i) - 2t}_{f(z(t))} \cdot \underbrace{(-3-i)}_{z'(t)} dt \\
 &\quad + \int_0^1 \underbrace{-2(1-t)}_{f(z(t))} \cdot \underbrace{2}_{z'(t)} dt \\
 &= \int_0^1 2t dt + \int_0^1 ((2i-4) + 10t) dt + \int_0^1 4(t-1) dt \\
 &= 1 + (2i-4) + 5 - 2 = 2i
 \end{aligned}$$

For part (b) we have

$$\int_0^{2\pi} \underbrace{i + 2e^{it}}_{f(z(t))} \cdot \underbrace{2ie^{it}}_{z'(t)} dt = \int_0^{2\pi} 2i(-i + 2e^{-it})e^{it} dt = 8\pi i.$$