P1// Let the map be $w = \dfrac{az+b}{cz+d} = T(z)$

$T(2) = 0 \Rightarrow 0 = \dfrac{2a+b}{2c+d} \Rightarrow 2a+b=0$ (i)

$T(i) = \infty \Rightarrow \lim_{z \to i} T(z) = \infty$

$\Leftrightarrow \lim_{z \to i} 1/T(z) = 0$

$\Leftrightarrow \lim_{z \to i} \dfrac{cz+d}{az+b} = 0$

$\Leftrightarrow \lim_{z \to i} cz+d = 0 \Rightarrow ci+d=0$ (ii)

& $T(0) = -2i \Leftrightarrow \dfrac{b}{d} = -2i$ (iii)

So choose e.g. $d=1$: (iii) $\Rightarrow b = -2i$

& (ii) $\Rightarrow ci+1=0$

$\Rightarrow c = +i$

So (i) $\Rightarrow 2a-2i=0 \Rightarrow a=i$

Map is $T(z) = \dfrac{iz-2i}{iz+1}$.

2. a) Let the Möbius Transformation be
$T(z) = \dfrac{az+b}{cz+d}$. Then $T(z)=z$ means

$$\frac{az+b}{cz+d} = z, \quad \text{i.e.}$$

$$cz^2+dz = az+b, \quad \text{i.e.}$$

$$cz^2+(d-a)z-b = 0 \quad (*)$$

If $c \neq 0$, $(*)$ is a quadratic and hence has one or two sol^{n}s, namely $\dfrac{-(d-a)+[(d-a)^2+4bc]^{1/2}}{2c}$.

If $c=0$, $(*)$ becomes

$$(d-a)z = b. \quad (**)$$

This has: one solution $\left(\frac{b}{d-a}\right)$ if $d \neq a$.

no solution if $d=a$ & $b \neq 0$

∞ly many sol^{n}s if $d=a$ & $b=0$.

However, in this last case, $b=c=0$, & so $ad-bc \neq 0$ means we must have $a=d\neq 0$.

In ~~that~~ case, we have $T(z) = \dfrac{az}{\cancel{bz}d} = z$,

i.e., the identity, which is excluded.

b)(i) From above, if $c \neq 0$ & $(d-a)^2+4bc \neq 0$, we will have 2 fixed points: e.g. $T(z) = 1/z$ (fixed pts $1, -1$)

(ii) e.g. $T(z) = z/2$ (fixed pt $= 0$).

(iii) e.g. $T(z) = z+1$.

③ a) $$\sin\bar{z} = \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} = \frac{\overline{e^{-iz}} - \overline{e^{iz}}}{\overline{-2i}} = \overline{\left(\frac{e^{-iz} - e^{iz}}{-2i}\right)} = \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} = \overline{\sin z}$$

b) note for $z = x+iy$:

$$\cosh\bar{z} = \cosh(x-iy) = \cos y + ix$$
$$= \cos y \cosh x - i \sin y \sinh x \quad (A)$$

& $$\cosh z = \cosh(x+iy) = \cos(-y+ix)$$
$$= \cos y \cosh x + i \sin y \sinh x \quad (B)$$

comparing (A) & (B) we see $\cosh\bar{z} = \overline{\cosh z}$.

Alternatively note

$$\cosh\bar{z} = \frac{e^{\bar{z}} + e^{-\bar{z}}}{2} = \overline{\left(\frac{e^{z} + e^{-z}}{2}\right)} = \overline{\cosh z},$$

because for $z = x+iy$, $e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = \overline{e^x e^{iy}}$
$= \overline{e^z}$.

4(a) $\operatorname{Log} z = 4i$ ①

$\Rightarrow \ln|z| + i\arg z = 4i$. ②

(remark: note the LHS of ① (or ②) is multi-valued in general, & the RHS is a single $\mathbb{C}$-number).

From ② we see, equating real & imaginary parts,

$$\begin{cases} \ln|z| = 0 & (i) \\ \arg z = 4 & (ii) \end{cases}$$

(note that, more formally, (ii) should be more like '$4 \in \arg z$').

From this we see the only sol^n is $z = \sin 4 + i\cos 4$.

Alternatively, you can argue via $z = \exp(\log z) = e^{4i}$.

④ b) $z^i = i$

$\Rightarrow \exp(i \log z) = i$

$\Rightarrow \exp[i(\ln|z| + i\arg z)] = \exp[i(\frac{\pi}{2} + 2j\pi)]$ $j \in \mathbb{Z}$

$\Rightarrow \arg z = 2k\pi \quad k \in \mathbb{Z}$

$\quad \& \ \ln|z| = \frac{\pi}{2} + 2j\pi \quad j \in \mathbb{Z}$

i.e., $z = \exp(\frac{\pi}{2} + 2j\pi) \quad j \in \mathbb{Z}$

⑤ a) $w = \tanh^{-1} z \Rightarrow z = \tanh w = \dfrac{\sinh w}{\cosh w}$

$$= \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}} = \frac{e^{2w} - 1}{e^{2w} + 1}$$

$$\Rightarrow (e^{2w} + 1) z = e^{2w} - 1$$

$$\Rightarrow \quad e^{2w} = \frac{z+1}{1-z} \quad z \neq 1$$

i.e. $\quad w = \frac{1}{2} \log \frac{1+z}{1-z}$ $z \neq 1, -1$ as req^d.

b) $\tanh^{-1} i = \frac{1}{2} \log \frac{1+i}{1-i}$

$$= \frac{1}{2} \log \frac{(1+i)^2}{(1+i)(1-i)}$$

$$= \frac{1}{2} \log \frac{2i}{2}$$

$$= \frac{1}{2} \log i$$

$$= \frac{i}{2}\left(\frac{\pi}{2} + 2n\pi\right) \quad n \in \mathbb{Z}$$

$$= i\left(\frac{\pi}{4} + n\pi\right) \quad n \in \mathbb{Z}$$

6a) Via the hint received on blackboard, we have:

$$\partial(\Omega_1 \cup \Omega_2) \subseteq \partial\Omega_1 \cup \partial\Omega_2 \quad (*).$$

Since Ω_1 & Ω_2 are even, there holds:

$$\partial\Omega_1 \subseteq \Omega_1 \quad \&$$

$$\partial\Omega_2 \subseteq \Omega_2.$$

hence $\partial\Omega_1 \cup \partial\Omega_2 \subseteq \Omega_1 \cup \Omega_2$ & so via $(*)$,

$$\partial(\Omega_1 \cup \Omega_2) \subseteq \Omega_1 \cup \Omega_2,$$

i.e., $\Omega_1 \cup \Omega_2$ is closed.

b) i) yes, e.g., $\Omega_1 = \overline{B}_1(0)$, $\Omega_2 = B_1(0)$:
$\Omega_1 \cup \Omega_2 = \overline{B}_1(0)$ is closed.

ii) no, e.g., $\Omega_1 = \overline{B}_1(0)$, $\Omega_2 = B_2(0)$:
$\Omega_1 \cup \Omega_2 = \Omega_2$ is open.