P1// Let the map be $w = \dfrac{az+b}{cz+d} = T(z)$

$T(2) = 0 \Rightarrow 0 = \dfrac{2a+b}{2c+d} \Rightarrow 2a+b=0$ (I)

$T(i) = \infty \Rightarrow \lim_{z\to i} T(z) = \infty$

$\Leftrightarrow \lim_{z \to i} 1/T(z) = 0$

$\Leftrightarrow \lim_{z\to i} \dfrac{cz+d}{az+b} = 0$

$\Leftrightarrow \lim_{z\to i} cz+d = 0 \Rightarrow ci+d=0$ (II)

& $T(0) = -2i \Leftrightarrow \dfrac{b}{d} = -2i$ (III)

So choose e.g. $d=1$: (III) $\Rightarrow b = -2i$

& (II) $\Rightarrow ci+1=0$

$\Rightarrow c = +i$

So (I) $\Rightarrow 2a - 2i = 0 \Rightarrow a = i$

Map is $T(z) = \dfrac{iz - 2i}{iz+1}$.

2. a) Let the Möbius Transformation be $T(z) = \dfrac{az+b}{cz+d}$. Then $T(z)=z$ means

$$\frac{az+b}{cz+d} = z, \quad \text{i.e.}$$

$$cz^2+dz = az+b, \quad \text{i.e.}$$

$$cz^2+(d-a)z-b = 0 \quad (*)$$

If $c \neq 0$, $(*)$ is a quadratic and hence has one or two sol^{n}s, namely $\dfrac{-(d-a)+[(d-a)^2+4bc]^{1/2}}{2c}$.

If $c=0$, $(*)$ becomes

$$(d-a)z = b. \quad (**)$$

This has: one solution $\left(\frac{b}{d-a}\right)$ if $d \neq a$.

no solution if $d=a$ & $b \neq 0$

∞ly many sol^{n}s if $d=a$ & $b=0$.

However, in this last case, $b=c=0$, & so $ad-bc \neq 0$ means we must have $a=d\neq 0$.

In ~~that~~ case, we have $T(z) = \dfrac{az}{d} = z$, i.e., the identity, which is excluded.

b)(i) From above, if $c \neq 0$ & $(d-a)^2+4bc \neq 0$, we will have 2 fixed points: e.g. $T(z) = 1/z$ (fixed pts $1,-1$)

(ii) e.g. $T(z) = z/2$ (fixed pt $= 0$).

(iii) e.g. $T(z) = z+1$.

③ a) $\sin\bar{z} = \dfrac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} = \dfrac{\overline{e^{-iz}} - \overline{e^{iz}}}{\overline{-2i}}$

$$= \overline{\left(\frac{e^{-iz} - e^{iz}}{-2i}\right)} = \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} = \overline{\sin z}$$

b) note for $z = x+iy$:

$$\cosh\bar{z} = \cosh(x-iy) = \cos y + ix$$
$$= \cos y \cosh x - i \sin y \sinh x \quad (A)$$
$$\& \ \cosh z = \cosh(x+iy) = \cos(-y+ix)$$
$$= \cos y \cosh x + i \sin y \sinh x \quad (B)$$

comparing (A) & (B) we see $\cosh\bar{z} = \overline{\cosh z}$.

Alternatively note

$$\cosh\bar{z} = \frac{e^{\bar{z}} + e^{-\bar{z}}}{2} = \overline{\left(\frac{e^{z} + e^{-z}}{2}\right)} = \overline{\cosh z},$$

because for $z = x+iy$, $e^{\bar{z}} = e^{x-iy} = e^{x}e^{-iy} = \overline{e^{x}e^{iy}}$
$= \overline{e^{z}}$.

4 a) $\log z = 4i \Rightarrow \exp(\log z) = e^{4i}$
$\Rightarrow \quad z = \cos 4 + i \sin 4$.

Alternatively, you can write
$\log z = 4i \Rightarrow \ln|z| + i \arg z = 4i$, so
$|z| = 1$ & $\arg z = 4 \Rightarrow z = \cos 4 + i \sin 4$.
(This is much easier than working in terms of arctan, which leads to $\arg z = \pi + \arctan(4-\pi)$, and ultimately to the same answer).

④ b) $z^i = i$

$\Rightarrow \exp(i \log z) = i$

$\Rightarrow \exp[i(\ln|z| + i \arg z)] = \exp[i(\pi/2 + 2j\pi)]$ $j \in \mathbb{Z}$

$\Rightarrow \arg z = 2k\pi \quad k \in \mathbb{Z}$

& $\ln|z| = \pi/2 + 2j\pi \quad j \in \mathbb{Z}$

i.e., $z = \exp(\pi/2 + 2j\pi) \quad j \in \mathbb{Z}$

P5// a) $w = \cot^{-1} z \Rightarrow z = \cot w = \dfrac{\cos w}{\sin w}$

$$= \frac{(e^{iw} + e^{-iw})/2}{(e^{iw} - e^{-iw})/2i}$$

$$= i \frac{(e^{2iw} + 1)}{e^{2iw} - 1}$$

$\Rightarrow (e^{2iw} - 1) z = e^{2iw} i + i$

i.e. $e^{2iw} = \dfrac{z+i}{z-i}$ $\qquad z \neq i, -i$

$\Rightarrow 2iw = \log\left(\dfrac{z+i}{z-i}\right)$

i.e. $w = \dfrac{-i}{2} \operatorname{Log}\left(\dfrac{z+i}{z-i}\right)$

b) $\cot z = 1 \Leftrightarrow z = \dfrac{-i}{2} \operatorname{Log}\left(\dfrac{1+i}{1-i}\right) = \dfrac{-i}{2} \operatorname{Log} i$

$= \dfrac{-i}{2}(\ln|i| + i \arg(i))$

$= \dfrac{-i}{2}(0 + (2k + 1/2)\pi i) \qquad k \in \mathbb{Z}$

$= (k + 1/4)\pi, \quad k \in \mathbb{Z}$.

6a) Via the hint received on blackboard, we have:

$$\partial(\Omega_1 \cup \Omega_2) \subseteq \partial\Omega_1 \cup \partial\Omega_2 \quad (*).$$

Since Ω_1 & Ω_2 are even, there holds:

$$\partial\Omega_1 \subseteq \Omega_1 \quad \&$$
$$\partial\Omega_2 \subseteq \Omega_2.$$

hence $\partial\Omega_1 \cup \partial\Omega_2 \subseteq \Omega_1 \cup \Omega_2$ & so via $(*)$,

$$\partial(\Omega_1 \cup \Omega_2) \subseteq \Omega_1 \cup \Omega_2,$$

i.e., $\Omega_1 \cup \Omega_2$ is closed.

b) i) yes, e.g., $\Omega_1 = \overline{B}_1(0)$, $\Omega_2 = B_1(0)$:
$\Omega_1 \cup \Omega_2 = \overline{B}_1(0)$ is closed.

ii) no, e.g., $\Omega_1 = \overline{B}_1(0)$, $\Omega_2 = B_2(0)$:
$\Omega_1 \cup \Omega_2 = \Omega_2$ is open.