COMPLEX ANALYSIS ASST. 3

1.a) $\lim_{z\to\infty} \frac{7z^3}{z^3-13z} = \lim_{z\to 0} \frac{7(1/z)^3}{(1/z)^3 - 13(1/z)}$

$= \lim_{z\to 0} \frac{7}{1-13z^2} = 7$.

b) $\lim_{z\to\infty} \frac{z^3}{z^2+13z} = \infty \iff \lim_{z\to 0}\left[\frac{1/z^3}{1/z^2 + 13/z}\right]^{-1} = 0$

$\iff \lim_{z\to 0}\left[\frac{1}{z+13z^2}\right]^{-1} = 0 \iff \lim_{z\to 0} z + 13z^2 = 0$,

which is true.

c) $\lim_{z\to\infty} \frac{(az+b)^2}{(cz+d)^2} = \lim_{z\to 0} \frac{(a/z+b)^2}{(c/z+d)^2}$

$= \lim_{z\to 0} \left(\frac{a+bz}{c+dz}\right)^2 = a^2/c^2$.

2. a) real and imaginary part are defined on all of $\mathbb{R}^2$ (polynomial functions), so function is defined on $\mathbb{C}$. Here $u(x,y) = 2xy$, $v(x,y) = x^2-y^2$,
So $u_x = 2y$, $u_y = 2x$, $v_x = 2x$, $v_y = 2y$.
(note: cts f^ns on $\mathbb{R}^2$)

C/R$_I$ $u_x = v_y \Rightarrow 2y = -2y \Rightarrow y = 0$.
C/R$_{II}$ $u_y = -v_x \Rightarrow 2x = -2x \Rightarrow x = 0$.

Hence C/R only hold at $(0,0)$. Hence f^n is only diff^{ble} at 0, so can't be analytic (not diff^{ble} on any nbhd of any pt).

b) $\bar{z} = x - iy$, so

$$\sin \bar{z} = \sin x \cosh(-y) + i\cos x \sinh(-y)$$
$$= \sin x \cosh y - i \cos x \sinh y \quad = u + iv$$

$\Rightarrow u_x = \cos x \cosh y$, $u_y = \sin x \sinh y$,
$v_x = \sin x \sinh y$, $v_y = -\cos x \cosh y$

C/R$_I$: $u_x = v_y \Leftrightarrow \cos x = 0$

C/R$_{II}$ $u_y = -v_x \Leftrightarrow \sin x = 0$ or $\sinh y = 0$

Hence C/R only hold at points of the form $((n+\tfrac{1}{2})\pi, 0)$. In particular, they don't hold on a nbhd of any pt. in $\mathbb{C}$ $\Rightarrow$ the f^n is nowhere analytic.

3. For this f^n, we have $u = x^4$, $v = (1-y)^4$
$\Rightarrow u_x = 4x^3$, $v_y = -4(1-y)^3$, $u_y = v_x = 0$.
So C/R$_I$ $\Rightarrow$ $4x^3 = -4(1-y)^3$, which holds iff $x^3 = -(1-y)^3 = (y-1)^3$, i.e., $x = y-1$, i.e., on the line $y = x+1$.

Note that u & v & partials are differentiable on $\mathbb{R}^2$.
~~Hence (b) f is differentiable precisely on the~~ line $y = x+1$ (C/R are necessary, so not diff^{ble} off the line: sufficient cond^{n}s satisfied on the line).
(a) f is not diff^{ble} on any nbhd of any pt, so not analytic anywhere.

4. (a) $u(x,y) = \sqrt{|xy|}$, $v(x,y) = 0$.

$\Rightarrow v_x = v_y = 0$.

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = 0.$$

$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = 0.$$

So C/R hold at $(0,0)$.

(b) Consider $\Delta z = h(1+i)$.
Then
$$\frac{f(0+\Delta z)-f(0)}{\Delta z} = \frac{\sqrt{|h\cdot h|}}{h(1+i)}$$
$$= \frac{|h|}{h}\cdot\frac{1}{1+i}$$
does not approach a limit as $h\to 0$ ($\to \frac{1}{1+i}$ as $h\to 0^+$, & $\to \frac{-1}{1+i}$ as $h\to 0^-$).
Hence $f'(0)$ can't exist.

(c) C/R is necessary but not sufficient for differentiability, so no contradiction.

(6)(a) Note $x_n, y_n \to 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8} = \frac{1}{1-(-\frac{1}{2})}$ (geom series) $= \frac{2}{3}$.

Hence: set $p = \frac{2}{3}+\frac{2}{3}i$, & assume $\Lambda \subset A \cup B$, where A, B open, disjoint, & $A \cap \Lambda \neq \emptyset$, $\Lambda \cap B \neq \emptyset$.
wlog $p \in B$.

Set $N = \min_{n \in \mathbb{N}} \{n : I_n \subset B\}$

Note that B is open, so $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(p) \subset B$, so $I_n \subset B$ $\forall n$ sufficiently large (since $x_n + iy_n \to p$). Hence N is well defined & finite, & indeed $N > 0$ (or else $\Lambda \subset B$).

Consider I_{N-1}, & set $q_t = t z_{N-1} + (1-t) z_N$, & define $\tau = \inf_{t \in [0,1]} \{t : q_s \in B\ \forall s \in [t,1]\}$

Note $q_1 = z_N \in B$ since $z_N \in I_N \subset B$, so τ is well defined.

$\tau = 1$ is not possible: since $z_N \in B \Rightarrow \exists$ some $\varepsilon' : B_{\varepsilon'}(z_N) \subset B \Rightarrow \tau < 1$.

$I_{N-1} \not\subset B \Rightarrow \tau > 0$.

So $\tau \in (0,1)$. Now consider q_τ. If $q_\tau \in B$, we know $\exists$ some $B_{\varepsilon'}(q_\tau) \subset B$, giving a contradiction to the defn of τ: if $q_\tau \in A$, we obtain a similar contradiction. Hence we can't find such A & B, & Λ is connected.

b) Any piecewise linear path in Λ connecting 0 to $\frac{2}{3} + \frac{2}{3}i$ must include an arbitrarily large number of the I_n's, so Λ is not affinely path connected in the sense given in class.

c) No contradiction. Piecewise affinely p.c. & connected are equivalent for open sets, but Λ is not open (can check easily that e.g. $0+0i$ is a boundary point; Λ is in fact closed).