

## COMPLEX ANALYSIS ASST 3

$$\textcircled{1} \quad a) \quad \lim_{z \rightarrow \infty} \frac{4z^5}{z^5 - 142z} = \lim_{z \rightarrow 0} \frac{4\left(\frac{1}{z}\right)^5}{\left(\frac{1}{z}\right)^5 - 142\left(\frac{1}{z}\right)}$$

$$= \lim_{z \rightarrow 0} \frac{4}{1 - 142z^4} = 4.$$

$$b) \quad \lim_{z \rightarrow \infty} \frac{z^4}{z^2 \cdot 142z} = \infty \Leftrightarrow \lim_{z \rightarrow 0} \left[ \frac{1/z^4}{1/z^2 + 142/z} \right]^{-1} = 0$$

$$\Leftrightarrow \lim_{z \rightarrow 0} \left[ \frac{1}{z^2 + 142z^3} \right]^{-1} = 0 \Leftrightarrow \lim_{z \rightarrow 0} (z^2 + 142z^3) = 0,$$

which is true.

$$c) \quad \lim_{z \rightarrow \infty} \frac{(az+b)^3}{(cz+d)^3} = \lim_{z \rightarrow 0} \frac{(a/z+b)^3}{(c/z+d)^3}$$

$$= \lim_{z \rightarrow 0} \left( \frac{a+bz}{c+dz} \right)^3 = \frac{a^3}{c^3}.$$

(2) a) real & imaginary parts ( $u = 2xy$ ,  $v = x^2 + y^2$ ) are defined on all of  $\mathbb{C}$ . Hence the function is defined on all of  $\mathbb{C}$ .

There holds:  $u_x = 2y$ ,  $u_y = 2x$ ,  $v_x = 2x$ ,  $v_y = 2y$ .  
So  $C/R_I \Rightarrow 2y = 2y$ , which holds on  $\mathbb{C}$ , but

$C/R_{II} \Rightarrow u_y = -v_x \Rightarrow 2x = -2x$ , which is only true on the imaginary axis. There is hence no point in  $\mathbb{C}$  for which the function is differentiable on a neighbourhood (since CLR are necessary for differentiability), & hence  $f^n$  is nowhere analytic.

b) Here the  $f^n$  is given by  $e^y e^{ix} = e^y (\cos x + i e^y \sin x)$ ,  
 So  $u_x = e^y \sin x$ ,  $u_y = e^y \cos x$ ,  
 $v_x = e^y \cos x$ ,  $v_y = e^y \sin x$  ( $f^n$  is defined on  $\mathbb{C}$ )  
 Hence  $C/R_I \Rightarrow -e^y \sin x = e^y \sin x \Leftrightarrow x = n\pi$ ,  
 and the remainder of the argument proceeds as in part a).

(3) For this  $f^n$ , we have  $u = x^3$ ,  $v = (1-y)^3$   
 $\Rightarrow u_x = 3x^2$ ,  $u_y = 0$ ,  $v_x = 0$ ,  $v_y = -3(1-y)^2$ .

So  $C/R_I \Rightarrow 3x^2 = -3(1-y)^2$ , which is only satisfied at  $x=0$ ,  $y=1$ .

$C/R_{II} \Rightarrow 0=0$  holds everywhere. Note also that  $u$  &  $v$  and all the first-order partials exist everywhere.

Since CLR only hold at  $i$  & CLR are necessary for differentiability, we see the  $f^n$  is only possibly diff<sup>ble</sup> at  $i$ . Hence in particular it is not diff<sup>ble</sup> on any nbhd of any pt., & hence is nowhere analytic.

b) By inspection  $v, u, u_x, u_y, v_x, v_y$  are cts on  $\mathbb{C}$ . Since CLR hold at  $i$ , this means  $f^n$  is indeed diff<sup>ble</sup> at  $i$ .

$$4. (a) u(x, y) = \sqrt{|xy|}, \quad v(x, y) = 0.$$

$$\Rightarrow v_x = v_y = 0.$$

$$u_x(0, 0) = \lim_{h \rightarrow 0} \frac{u(h, 0) - u(0, 0)}{h} = 0.$$

$$u_y(0, 0) = \lim_{h \rightarrow 0} \frac{u(0, h) - u(0, 0)}{h} = 0.$$

So C/R hold at  $(0, 0)$ .

(b) Consider  $\Delta z = h(1+i)$ .

$$\text{Then } \frac{f(0+\Delta z) - f(0)}{\Delta z} = \frac{\sqrt{|h \cdot h|}}{h(1+i)}$$

$$= \frac{|h|}{h} \cdot \frac{1}{1+i} \text{ does not approach a}$$

limit as  $h \rightarrow 0$  ( $\rightarrow \frac{1}{1+i}$  as  $h \rightarrow 0^+$ ,  $\Delta$

$\rightarrow \frac{-1}{1+i}$  as  $h \rightarrow 0^-$ ).

Hence  $f'(0)$  can't exist.

(c) C/R is necessary but not sufficient for differentiability, so no contradiction.

⑥ Note  $x_n, y_n \rightarrow 1^{-\frac{1}{2}} + \frac{1}{4} - \frac{1}{8} = \frac{1}{1(\frac{1}{2})}$  (geom series)  $= \frac{2}{3}$ .

Hence: set  $p = \frac{2}{3} + \frac{2}{3}i$ , & assume  $\Lambda \subset A \cup B$ , where  $A, B$  open, disjoint, &  $A \cap \Lambda \neq \emptyset, \Lambda \cap B \neq \emptyset$ . wlog  $p \in B$ .

Set  $N = \min_{n \in \mathbb{N}} \{n : I_n \subset B\}$

Note that  $B$  is open, so  $\exists \varepsilon > 0$  s.t.  $B_\varepsilon(p) \subset B$ , so  $I_n \subset B \forall n$  sufficiently large (since  $x_n + iy_n \rightarrow p$ ). Hence  $N$  is well defined & finite, & indeed  $N > 0$  (or else  $\Lambda \subset B$ ).

Consider  $I_{N-1}$ , & set  $q_t = t z_{N-1} + (1-t) z_N$ , & define  $\tau = \inf_{t \in [0,1]} \{t : q_s \in B \forall s \in [t, 1]\}$

Note  $q_1 = z_N \in B$  since  $z_N \in I_N \subset B$ , so  $\tau$  is well defined.

~~$\tau = 1$~~  is not possible: since  $z_N \in B \Rightarrow \exists$  some  $\varepsilon' : B_{\varepsilon'}(z_N) \subset B \Rightarrow \tau < 1$ .

$I_{N-1} \not\subset B \Rightarrow \tau > 0$ .

So  $\tau \in (0, 1)$ . Now consider  $q_\tau$ . If  $q_\tau \in B$ , we know  $\exists$  some  $B_{\varepsilon'}(q_\tau) \subset B$ , giving a contradiction to the def<sup>n</sup> of  $\tau$ : if  $q_\tau \in A$ , we obtain a similar contradiction. Hence we can't find s.t.  $A \& B$ , &  $\Lambda$  is connected.

b) Any piecewise linear path in  $\Lambda$  connecting  $0$  to  $\frac{2}{3} + \frac{2}{3}i$  must include an arbitrarily large number of the  $I_n$ 's, so  $\Lambda$  is not affinely path connected in the sense given in class.

c) No contradiction. Piecewise affinely p.c. & connected are equivalent for open sets, but  $\Lambda$  is not open (can check easily that e.g.  $0 + 0i$  is a boundary point:  $\Lambda$  is in fact closed).