MATH39.01 Asst 4.

① $f(z) = \sin z$ is analytic on $\mathbb{C}$, so by Cauchy's formula, $f^{(v)}(-1) = \frac{5!}{2\pi i}\int_C \frac{f(z)}{(z+1)^{6+1}}dz$

i.e.
$$\int_C \frac{\sin z\, dz}{(z+1)^7} = \frac{2\pi i f^{(vi)}(-1)}{6!} = \frac{2\pi i(-\sin(-1))}{6!} = \frac{2\pi i}{6!}\sin 1 = \frac{\pi i \sin 1}{360}.$$

② a) $u = e^x\cos y \Rightarrow u_x = e^x \cos y,\ u_{xx} = e^x\cos y;\ u_y = -e^x \sin y,\ u_{yy} = -e^x\cos y$

$\Rightarrow \Delta u = u_{xx} + u_{yy} = 0.$ (note $u \in C^\infty$)

conj. harm f^n v: u & v solve $\begin{cases} u_x = v_y & ① \\ u_y = -v_x & ② \end{cases}$

$① \Rightarrow v_y = e^x \cos y$

$\Rightarrow v = \int e^x \cos y\, dy + \Phi(x)$ $= e^x \sin y + \Phi(x)$

$\Rightarrow v_x = e^x \sin y + \Phi'(x) \overset{②}{=} -u_y = e^x \sin y$

$\Rightarrow \Phi' = 0 \Rightarrow \Phi = C$, so for e.g. $C = 0$, $v = e^x \sin y$ is a harm conj of u. (Note then $u + iv = e^x e^{iy} = e^{x+iy}$ So u & v are the Re & Im parts of $f(z) = e^z$).

b) $u = x^2 - y^2 - 2y \Rightarrow u_x = 2x,\ u_{xx} = 2;\ u_y = -2y - 2,\ u_{yy} = -2$

$\Rightarrow \Delta u = 0$ (note $u \in C^\infty$).

conj harm f^n v: u & v solve $\begin{cases} u_x = v_y & ① \\ u_y = -v_x & ② \end{cases}$

$u_x = 2x \overset{①}{=} v_y \Rightarrow v = 2xy + \Phi(x)$

$\Rightarrow v_x = 2y + \Phi'(x) = -u_y = 2y + 2$

$\Rightarrow \Phi'(x) = 2 \Rightarrow \Phi = 2x + C$ e.g. $C = 0$, $v = 2xy + 2x$ is a harm conj.

Note then $f(z) = u + iv = (x^2 - y^2 - 2y) + (2xy + 2x)i$
$= z^2 + 2iz.$

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3) a) Let $\underline{\underline{\nu}}$ be the external unit normal to Ω.
The Neumann b.v.p. for Laplace's eq^{n} is

$$(N)\quad \begin{cases} \Delta f = 0 & \text{in } \Omega \quad (N1) \\ \dfrac{df}{d\underline{\underline{\nu}}} = g & \text{on } \partial\Omega \quad (N2) \end{cases}$$

If u solves (N), then $\Delta(u+c) = \Delta u \overset{(N1)}{=} 0$, so
$u+c$ solves (N1). Further,

$$\frac{d}{d\underline{\underline{\nu}}}(u+c) = \nabla(u+c)\cdot\underline{\underline{\nu}}$$
$$= \nabla u\cdot\underline{\underline{\nu}}$$
$$= \frac{du}{d\underline{\underline{\nu}}} \overset{(N2)}{=} g,$$

so $u+c$ also solves (N2). Hence $u+c$ solves (N).

b) The Dirichlet b.v.p. for Laplace's eq^{n} is

$$(D)\quad \begin{cases} \Delta f = 0 & \text{in } \Omega \quad (D1) \\ f = \varphi & \text{on } \partial\Omega, \quad (D2) \end{cases}$$

where φ is given.
If u solves (D), then certainly (D1) will be solved for any $c \in \mathbb{R}$: however, (D2) will only be solved for $c=0$. Hence, no.

(4) $$\frac{1}{4-z} = \frac{1}{4-3i-(z-3i)}$$

$$= \frac{1}{4-3i}\,\frac{1}{\left(1-\frac{z-3i}{4-3i}\right)}$$

$$= \frac{1}{4-3i}\sum_{n=0}^{\infty}\left(\frac{z-3i}{4-3i}\right)^n \quad \text{for } \left|\frac{z-3i}{4-3i}\right|<1,$$

i.e., for $|z-3i| < |4-3i| = 5$

$$= \sum_{n=0}^{\infty}\frac{(z-3i)^n}{(4-3i)^{n+1}}$$, with radius of convergence

5, noting

$$\Lambda = \lim_{n\to\infty}\frac{1/|4-3i|^{n+2}}{1/|4-3i|^{n+1}} = \lim_{n\to\infty}\frac{1}{|4-3i|} = \frac{1}{5}.$$

(Alternatively, note series will converge out to the first singularity, which is at 4, & $|4-3i| = 5$).

(5) For $f(z) = \sinh z$, we have

$$f^{(n)}(z) = \begin{cases} \sinh z & z \text{ even} \\ \cosh z & z \text{ odd} \end{cases}$$

$$\Rightarrow f^{(n)}(0) = \begin{cases} \cosh 0 = 1 & n \text{ odd} \\ \sinh 0 = 0 & n \text{ even} \end{cases}$$

$\Rightarrow$ Maclaurin series for $\sinh z$ is

$$z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$\Rightarrow$ Laurent series for $\sinh z^{-1}$ is

$$\frac{1}{z} + \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} + \cdots$$

$\Rightarrow$ Laurent series for $z^{-1}\sinh(z^{-1})$ is

$$\frac{1}{z^2} + \frac{1}{3!\,z^4} + \frac{1}{5!\,z^6} + \cdots \qquad (L)$$

Note that $z^{-1}\sinh(z^{-1})$ is analytic for $z \neq 0$, so the origin is an isolated singularity. From (L), there are arbitrarily large negative powers of z in this, so 0 is an essential singularity.

$h \neq 0$

(6) a) Note for $z = h + 0i$, $f(z) = h^5/|h|^4 = h + 0i$

& for $z = 0 + hi$, $f(z) = h^5 i/h^4 = 0 + hi$.

Hence $u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$

$= \lim_{h\to 0} h/h = 1.$

$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h}$

$= \lim_{h\to 0} \frac{0-0}{h} = 0.$

$v_y(0,0) = \lim_{h\to 0} \frac{v(0,h)-v(0,0)}{h}$

$= \lim_{h\to 0} h/h = 1$

& $u_y(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h}$

$= \lim_{h\to 0} \frac{0-0}{h} = 0.$

Hence, C/R at $(0,0)$ hold $(1=1, 0=0)$.

b) Approach along the x axis:

$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0}\left[\frac{h^5/h^4 - 0}{h}\right] = 1$

Approach along the line $y=x$:

$\lim_{h\to 0} \frac{f(h(1+i))-f(0)}{h(1+i)} = \lim_{h\to 0}\left[\left(\frac{h^5(1+i)^5}{4h^4} - 0\right)\Big/ h(1+i)\right]$

$= \lim_{h\to 0} \frac{(1+i)^4}{4} = -1.$

Derivative must be independent of direction, so can't exist.

c) C/R is necessary but not sufficient for differentiability.

(7) a) Let $S = \sum_{n=0}^{\infty} c_n z^n$ for z such that the series converges.

Then $zS = \sum_{n=0}^{\infty} c_n z^{n+1} = \sum_{n=1}^{\infty} c_{n-1} z^n$

& $z^2 S = \sum_{n=0}^{\infty} c_n z^{n+2} = \sum_{n=2}^{\infty} c_{n-2} z^n$

So, $S - zS - z^2 S = c_0 + c_1 z + \sum_{n=2}^{\infty} c_n z^n - c_0 - \sum_{n=2}^{\infty} c_{n-1} z^n - \sum_{n=2}^{\infty} c_{n-2} z^n$ (*)

But $\frac{1}{1-z-z^2} = S \Rightarrow$ LHS of (*) $= 1$.

So, equating coefficients, we have:

$c_0 = 1$,

$c_1 - c_0 = 0 \Rightarrow c_1 = 1$,

$c_n - c_{n-1} - c_{n-2} = 0$ for $n \geq 2 \Rightarrow c_n = c_{n-1} + c_{n-2}$, as req^d.

b) Series will converge out to the first singularity, i.e., sol^n of $z(z+1) = 0$, i.e., $\frac{-1 \pm \sqrt{5}}{2}$.

Of these, $\frac{\sqrt{5}-1}{2}$ is closest to 0: so

$R = \left| \frac{\sqrt{5}-1}{2} \right| = \frac{\sqrt{5}-1}{2}$.

c) Fibonacci ~~A~~S would be a good name.