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① a) Note for $z = h + 0i$, $h \neq 0$, $f(z) = h^5/|h|^4 = h + 0i$,
& for $z = 0 + hi$, $f(z) = h^5 i / h^4 = 0 + hi$.

Hence,
$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h};$$
$$= \lim_{h \to 0} h/h = 1$$
$$v_x(0,0) = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0-0}{h} = 0;$$
$$u_y(0,0) = \lim_{h \to 0} \frac{u(0,h) - u(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0-0}{h};$$
$$\& \quad v_y(0,0) = \lim_{h \to 0} \frac{v(0,h) - v(0,0)}{h}$$
$$= \lim_{h \to 0} h/h = 1.$$

Hence, C/R hold at $(0,0)$ $(1 = 1, \; -0 = 0)$

b) approach along x axis: $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \left[\frac{h^5/h^4 - 0}{h} \right] = 1.$

approach along the line $y = x$:
$$\lim_{h \to 0} \frac{f(h(1+i)) - f(0)}{h(1+i)} = \lim_{h \to 0} \frac{\left(h^5(1+i)^5 / 4h^4\right) - 0}{h(1+i)} = \lim_{h \to 0} \frac{(1+i)^4}{4} = -1$$

Derivative must be independent of direction, so can't exist.

c) C/R are necessary but not sufficient for differentiability.

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(2) a) $u = e^x \cos y \Rightarrow u_x = e^x \cos y$, $u_{xx} = e^x \cos y$; $u_y = -e^x \sin y$, $u_{yy} = -e^x \cos y$

$\Rightarrow \Delta u = u_{xx} + u_{yy} = 0$. (note $u \in C^\infty$)

conj. harm f^n v: u & v solve $\begin{cases} u_x = v_y & (1) \\ u_y = -v_x & (2) \end{cases}$

(1) $\Rightarrow v_y = e^x \cos y$

$\Rightarrow v = \int e^x \cos y \, dy + \Phi(x)$ $= e^x \sin y + \Phi(x)$

$\Rightarrow v_x = e^x \sin y + \Phi'(x) \overset{(2)}{=} -u_y = e^x \sin y$

$\Rightarrow \Phi' = 0 \Rightarrow \Phi = C$, so for e.g. $C = 0$, $v = e^x \sin y$ is a harm conj of u. (Note then $u + iv = e^x e^{iy} = e^{x+iy}$ So u & v are the Re & Im parts of $f(z) = e^z$).

b) $u = x^2 - y^2 - 2y \Rightarrow u_x = 2x$, $u_{xx} = 2$; $u_y = -2y - 2$, $u_{yy} = -2$

$\Rightarrow \Delta u = 0$ (note $u \in C^\infty$).

conj harm f^n v: u & v solve $\begin{cases} u_x = v_y & (1) \\ u_y = -v_x & (2) \end{cases}$

$u_x = 2x \overset{(1)}{=} v_y \Rightarrow v = 2xy + \Phi(x)$

$\Rightarrow v_x = 2y + \Phi'(x) = -u_y = 2y + 2$

$\Rightarrow \Phi'(x) = 2 \Rightarrow \Phi = 2x + C$ e.g. $C = 0$, $v = 2xy + 2x$ is a harm conj

Note then $f(z) = u + iv = (x^2 - y^2 - 2y) + (2xy + 2x)i$

$= z^2 + 2iz$.

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3 a) Let $\underline{\nu}$ be the external unit normal to Ω.
The Neumann b.v.p. for Laplace's eqⁿ is

$$(N)\quad \begin{cases} \Delta f = 0 & \text{in } \Omega \quad (N1) \\ \dfrac{df}{d\underline{\nu}} = g & \text{on } \partial\Omega \quad (N2) \end{cases}$$

If u solves (N), then $\Delta(u+c) = \Delta u \overset{(N2)}{=} 0$, so $u+c$ solves (N1). Further,

$$\begin{aligned} \frac{d}{d\underline{\nu}}(u+c) &= \nabla(u+c)\cdot\underline{\nu} \\ &= \nabla u \cdot \underline{\nu} \\ &= \frac{du}{d\underline{\nu}} \overset{(N2)}{=} g, \end{aligned}$$

so $u+c$ also solves (N2). Hence $u+c$ solves (N).

b) The Dirichlet b.v.p. for Laplace's eqⁿ is

$$(D)\quad \begin{cases} \Delta f = 0 & \text{in } \Omega \quad (D1) \\ f = \varphi & \text{on } \partial\Omega, \quad (D2) \end{cases}$$

where φ is given.
If u solves (D), then certainly (D1) will be solved for any $c \in \mathbb{R}$: however, (D2) will only be solved for $c=0$. Hence, no.

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(4) $$\frac{1}{3-z} = \frac{1}{3-4i-(z-4i)}$$
$$= \frac{1}{3-4i}\;\frac{1}{1-\left[{z-4i}/{(3-4i)}\right]}$$
$$= \frac{1}{3-4i}\sum_{n=0}^{\infty}\left(\frac{z-4i}{3-4i}\right)^n \quad \text{for } \left|\frac{z-4i}{3-4i}\right| < 1,$$

i.e. for $|z-4i| < |3-4i| = 5$

$$= \sum_{n=0}^{\infty}\frac{(z-4i)^n}{(3-4i)^{n+1}},$$ with radius of convergence $= 5$, since

$$\Lambda = \lim_{n\to\infty}\frac{1/|3-4i|^{n+2}}{1/|3-4i|^{n+1}} = \lim_{n\to\infty}\frac{1}{|3-4i|} = \frac{1}{5}$$
$$\Rightarrow R = 5.$$

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(5) For $f(z) = \sinh z$, we have

$$f^{(n)}(z) = \begin{cases} \sinh z & z \text{ even} \\ \cosh z & z \text{ odd} \end{cases}$$

$$\Rightarrow f^{(n)}(0) = \begin{cases} \cosh 0 = 1 & n \text{ odd} \\ \sinh 0 = 0 & n \text{ even} \end{cases}$$

$\Rightarrow$ Maclaurin series for $\sinh z$ is

$$z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$\Rightarrow$ Laurent series for $\sinh z^{-1}$ is

$$\frac{1}{z} + \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} + \cdots$$

$\Rightarrow$ Laurent series for $z^{-1}\sinh(z^{-1})$ is

$$\frac{1}{z^2} + \frac{1}{3!\,z^4} + \frac{1}{5!\,z^6} + \cdots \qquad (L)$$

Note that $z^{-1}\sinh(z^{-1})$ is analytic for $z \neq 0$, so the origin is an isolated singularity. From (L), there are arbitrarily large negative powers of z in this, so 0 is an essential singularity.

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(6) Put $z = re^{i\theta}$. $|z| < 1 \Rightarrow$

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad (**)$$

$$\text{LHS of } (**) = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta$$

by de Moivre.

$$\text{RHS of } (**) = \frac{r\cos\theta + i\sin\theta}{(1-r\cos\theta) - ir\sin\theta} \cdot \frac{(1-r\cos\theta) + ir\sin\theta}{(1-r\cos\theta) + ir\sin\theta}$$

$$= \frac{r\cos\theta - r^2\cos^2\theta - r^2\sin^2\theta + i(r\sin\theta - r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta)}{(1-r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \frac{r\cos\theta - r^2 + ir\sin\theta}{1 - 2r\cos\theta + r^2}.$$ using $\sin^2\theta + \cos^2\theta = 1$

Taking real & imaginary components yields the result.

⑦ a) Let $S = \sum_{n=0}^{\infty} c_n z^n$ for z such that the series converges.

Then $zS = \sum_{n=0}^{\infty} c_n z^{n+1} = \sum_{n=1}^{\infty} c_{n-1} z^n$

& $z^2 S = \sum_{n=0}^{\infty} c_n z^{n+2} = \sum_{n=2}^{\infty} c_{n-2} z^n$

So, $S - zS - z^2 S = c_0 + c_1 z + \sum_{n=2}^{\infty} c_n z^n - c_0 - \sum_{n=2}^{\infty} c_{n-1} z^n$
$- \sum_{n=2}^{\infty} c_{n-2} z^n$ ⊛

But $\frac{1}{1-z-z^2} = S \Rightarrow$ LHS of ⊛ $= 1$.

So, equating coefficients, we have:

$c_0 = 1$,

$c_1 - c_0 = 0 \Rightarrow c_1 = 1$,

$c_n - c_{n-1} - c_{n-2} = 0$ for $n \geq 2 \Rightarrow c_n = c_{n-1} + c_{n-2}$, as req^d.

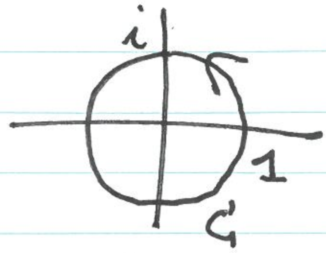
b) Series will converge out to the first singularity, i.e., sol^n of $z(z+1) = 0$, i.e., $\frac{-1 \pm \sqrt{5}}{2}$.

Of these, $\frac{\sqrt{5}-1}{2}$ is closest to 0: so

$R = \left|\frac{\sqrt{5}-1}{2}\right| = \frac{\sqrt{5}-1}{2}$.

c) Fibonacci ~~#~~s would be a good name.

⑧ $I = \int_0^{2\pi} \cos^{2n}\theta \, d\theta$



Put $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} \Rightarrow d\theta = \dfrac{dz}{iz}$,

& $\cos\theta = z + z^{-1}$

So, $I = \dfrac{1}{2^{2n} i} \displaystyle\int_{C_1} \frac{(z+z^{-1})^{2n}}{z} dz = \frac{1}{2^{2n} i} \int_{C_1} \sum_{k=0}^{n} \binom{2n}{k} z^{2n-k} (z^{-1})^k z^{-1} dz$

$$= \frac{1}{2^{2n} i} \int_{C_1} \sum_{k=0}^{n} \binom{2n}{k} z^{2n-2k-1} dz .$$

By Cauchy, all terms vanish except for $2n-2k-1 = -1$, i.e., $k=n$.

Hence, $I = \dfrac{1}{2^{2n} i} \dbinom{2n}{n} \displaystyle\int_C z^{-1} dz$

$$= \frac{1}{2^{2n} i} \frac{(2n)!}{n!\, n!} \cdot 2\pi i$$

$$= 2\pi \frac{\cancel{(2n)}(2n-1)\cancel{(2n-2)} \cdot \ldots \cdot 3 \cdot \cancel{2} \cdot 1}{\cancel{2 \cdot 4 \cdot 6 \ldots (2n)} \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$$

$$= 2\pi \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$$ as req^d.