A fn is **entire** if it is **analytic** on all of C, e.g., polynomials, exp, sin, cos, sinh, cosh...

Note: if a fn is diff'ble at precisely one pt, it is not analytic there, or any-

where. Such fn’s exist (e.g. \(|z|^2\)).
\[
\frac{d}{dz} \left( \log z \right); \quad |z| > 0
\]

Recall \( \log z = \ln |z| + i \arg z \)

\[
\begin{align*}
\text{Re} \ z & = \ln |z| + i \theta \\
\Rightarrow M & = \ln r, \quad V = 0 \\
ur & = \frac{1}{r}, \quad u_\theta = 0, \quad v_r = 0, \quad v_\theta = 1.
\end{align*}
\]

C/R in polar coords (Lec. 14)

\[
\alpha \times 6\times 2
\]

\[
ru_r = v_\theta \checkmark \\
u_\theta = -rv_r \checkmark
\]

Satisfies suff. cond. for (complex) diff. on any subset of \( \mathbb{C} \)

\[
\alpha \geq \theta = \alpha + 2\pi
\]

\[
\text{So, if we specify branch, we that log is} \]

\[
\frac{d}{dz} \left( \log z \right) = e^z
\]

\[
\alpha \text{ fixed in IR}
\]
So, if we specify a branch, we have that log is defined when 0 ≤ Arg z ≤ π.

\[ \frac{d}{dz} \log z = \frac{e^{-i\theta(z)}}{z} = \frac{e^{-i\arg(z)}}{z} = \frac{1}{re^{i\theta}} = \frac{1}{z}. \]

\[ e.g. \frac{d}{dz} \log z = \frac{1}{z} \text{ for } -\pi < \text{Arg } z < \pi, \quad |z| > 0. \]

* for \( f(z) = z^c, \quad c \in \mathbb{C} \) fixed defined on \( \mathbb{C}^* \):

\[ f(z) = \exp(c \log z), \quad f'(z) = \exp(c \log z) \frac{z}{z} \]

\[ \text{due to needing a branch of } \]
\[ e^{-c^2} = e^{-\frac{1}{2} 
abla^2} \]

\[ \bigcirc \text{ is valid on any set } \Omega \text{ f. } \]
\[ \exists \varepsilon > 0, \alpha < \arg 2 < \frac{3\pi}{2} \]

(due to needing to pick a branch of $\log \in \bigcirc$).

*RMK: $c = 0$

*RMK: Look for $c = 0$

*RMK: Try for $\arg(z) = c^2$. 

\[ \bigcirc \text{ pick in } \bigcirc \]
Notation from real analysis

\[ \Omega = \mathbb{R}^n \]

* \( C(\Omega) = C^0(\Omega) = \{ f: \Omega \to \mathbb{R} \mid f \text{ is continuous} \} \)

* \( C^k(\Omega) = \{ f: \Omega \to \mathbb{R} \mid f \text{ is } k \text{ times continuously differentiable} \} \)

\( f \in C^w(\Omega) \) says, at every \( x_0 \in \Omega \):

(a) \( f \) has a power series expansion at \( x_0 \) (namely)
its Taylor series); if \( f \) is given by its power series expansion i.e., power series converges to the function on some nbd of \( x \), a.k.a. real analytic f's

\[
\text{Note (vi) } \Rightarrow f \in \mathcal{C}^a(\Omega).
\]

\[
\text{Note (vi) } \not\Rightarrow (vi) \text{ in } \mathbb{R}^n
\]

Consider
\[
f(x) = \begin{cases} x^2 & x < 0 \\ 0 & x \geq 0 \end{cases}
\]

Check: \( f^{(n)}(x) \) exists \( A \neq 0 \);
\( f^{(n)}(0) = 0 \forall n \Rightarrow f^{(n)} \text{ is cts on } \mathbb{R} \)

\( \Rightarrow \text{Taylor} 0 \text{ is } \in \mathcal{C}^a \)
$\Rightarrow$ Taylor series of $f$ at $0$ is
\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0. \]

$f \in C^\infty(\mathbb{R})$, $f \notin C^\infty(\mathbb{R})$

$next: integration$