Define \( \tilde{z}(z) = \frac{z}{|z|^2} \) on \( \mathbb{C}_* \)

\[
\Delta \tilde{\eta}(z) = \tilde{z} \quad \text{on} \quad \mathbb{C}
\]

So \( \eta(\tilde{z}(z)) = \left( \frac{z}{|z|^2} \right) \)

\[
\tilde{z} = \frac{z_3}{|z_3|^2} = \frac{z_3}{|z_3|^2} = \frac{z_3}{|z_3|^2} = \frac{1}{2}
\]

\( \tilde{z} \) is called inversion (w.r.t. the unit circle) \( \eta \) in reflection in the real axis.
For \( w = 1/2 = \frac{\pi}{12} \): 

\[
x + iy = u + iv \\
w = \frac{x-iy}{x^2 + y^2} \Rightarrow u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}
\]

Use this to show:

\( \frac{\pi}{12} \) maps circles & lines in the \( z \)-plane to circles & lines in the \( w \)-plane

**Note:** Circles & lines in the \( z \)-plane can be represented as:

[Terminology]

\[ A(x^2 + y^2) + Bx + Cy + D = 0 \]

\[ A, B, C, D \in \mathbb{R}, \quad B^2 + C^2 > 4AD \]

\( (A = 0 \Rightarrow \text{line}, \quad A \neq 0 \Rightarrow \text{circle}) \)

Check for \( f = \frac{1}{2} \):

\[ D(u^2 + v^2) + Bu - Cv + A = 0 \]
Terminology

\[ f: \mathbb{R} \to \mathbb{C} \]

*injective* say:

\[ f(x) = f(y) \implies x = y. \]

*onto* or *surjective* say:

\[ A = 0. \]

Given \( \mathbb{N} \in \mathbb{Z} \) (at least one) \( z \in \mathbb{Z} : f(z) = y \).

Möbius transformations

(\( \beta - c \) \( \neq 0 \), \( \beta e - d \neq 0 \))

Def. Let \( a, b, c, d \in \mathbb{C} \) with \( ad - bc \neq 0 \)

Then:
\[ w = T(z) = \frac{az+b}{cz+d} \]  

is a Möbius (a.k.a. linear fractional) transformation.

Natural domain of def:

\* \( c=0 \) : \( \text{dom}(f) = \mathbb{C} \)

\* \( c \neq 0 \) : \( \text{dom}(f) = \mathbb{C} \setminus \frac{d}{c} \)

Note: \( \mathbb{C} \) can be rewritten as

\[ Azw + Bz + Cw + D = 0 \]  

(with \( A = c, B = -a, C = d \) & \( D = b \)) called implicit form.

Goal: understand \( T \).

CASE I: \( c = 0 \) (\( 0 \Rightarrow ad \neq 0 \))
Claim: $T$ maps $C \to C$ $1 \sim 1 \& \text{onto}$.

Pf: $1 \sim 1$: suppose $T(z) = T(w) \implies z = w$.

WTS: $z = \frac{w}{a}$.

\[
\frac{a}{d} \frac{a+b}{d} = \frac{a \frac{a+b}{d} + b}{d}
\]

(subtract by $d$, multiply by $\frac{1}{d}$)

$\implies z = \frac{w}{a}$

(ii) Given $w \in C$ need $z \in C$.

S.t. $T(z) = w$.

Check: $z = \frac{d}{a}(w - b) \text{ works.}$

CASE II $c \neq 0$.

\[
W = \frac{az+b}{cz+d} = \frac{a(z+\frac{b}{c}) - \frac{ad}{c} + b}{c(z+\frac{b}{c})}
\]

$\implies z = \frac{a}{c} + \frac{bc-ad}{c^2+d}$.
In this case, $T$ is a composition of the mappings:

- $z = c^2 + d$;
- $W = \frac{1}{2}$;
- $W = \frac{a}{c} + \frac{bc-ad}{c}$.

In case I, the equation is:

- $z = \frac{-dW+b}{cW-a}$.

In case II, the equation is:

- $z = \frac{-dW+b}{cW-a}$.

Note in case II:

- $T^{-1}(w) = \frac{-dW+b}{cW-a}$. 

In both cases I and II, $T$ is a composition of maps previously studied.
T is 1-1 & onto:
\[ C \setminus \{ p, q, r \} \rightarrow C \setminus \{ p, q, r \} \]