LECTURE 2

Further motivation:

* (i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \pi^2/6$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\cdots = \pi^2/12$

(iii) $\sum_{k=1}^{\infty}\frac{1}{1+4k^2\pi^2} = \frac{1}{2}\left(\frac{1}{e-1}-\frac{1}{2}\right).$

Re (i) $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$: Riemann zeta f^n : $s\in\mathbb{C}$.

\zeta

Riemann hypothesis: ζ has ∞ly many non-trivial zeros, & they all lie on the line $\mathrm{Re}(s)=\frac{1}{2}$.

Note that the def^n as given only makes sense for $\mathrm{Re}(s)>1$, so we need to extend ζ to $\mathbb{C}$, which we do by analytic continuation.

* trivial zeros $-2,-4,-6,\ldots$.
* Millenium Problems.

Complex Numbers:

* **1545** Cardano (Ars Magna)
  real roots of x^3+ax+b
  $5+\sqrt{-15}$ "mental torture".
* **1575** i: rules for operations in $\mathbb{C}$.
* **1629** $a+\sqrt{-b}$ "solutions impossibles"
  Decartes ~ imaginary numbers.
* **1831** complex #s.

INTRO BC §1-3

$\mathbb{N}$: $\{1,2,3,\dots\}$ natural numbers.

$\mathbb{N}_0$: $\{0,1,2,3,\dots\}$.

$\mathbb{Z}$: $\{0,\pm1,\pm2,\dots\}$ integers.

$\mathbb{Q}$: $\{p/q : p,q \in \mathbb{Z}, q \neq 0\}$.

$\mathbb{R}$: real #s.

$\mathbb{C}$: complex #s.

Various equivalent representations of $\mathbb{C}$.

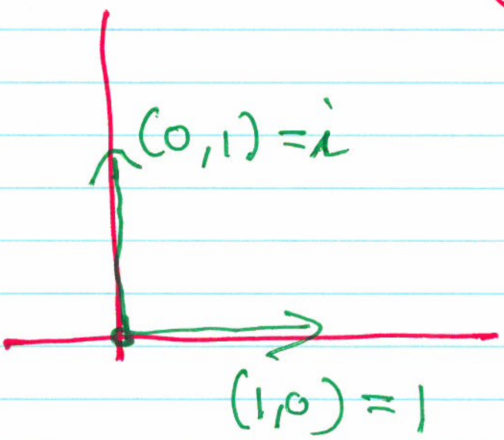
$z \in (\underset{\mathbb{R}}{x}, \underset{\mathbb{R}}{y}) \in$ complex plane.

So, $z = x(1,0) + y(0,1)$.

$$= x + iy$$

real part of z

imaginary part of z.



"i is the complex number represented by $(0,1)$".

We say $\mathbb{R} \subset \mathbb{C}$ by identifying the complex # $\boxed{x+0i}$ with the real number $\boxed{x}$.

Addition in $\mathbb{C}$: $+$

Set $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, i.e.

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Multiplication in $\mathbb{C}$:

$\times$ or $\cdot$ or juxtaposition:

$$(x_1, y_1)\cdot(x_2, y_2) = (x_1x_2 - y_1y_2,\ y_1x_2 + x_1y_2),$$

i.e. $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$.

Note: the definition of $\times$ formally applies if we use the usual rules for multiplication/ ~~in~~ algebra in $\mathbb{R}$, & set $i^2 = -1$.

$\boxed{\text{With this } + \,\&\, \times,\ \mathbb{C} \text{ is a field!}}$

Check! $\mathbb{C}$ is closed under $+$ & $\times$.

F2) (i) $0 = 0 + 0i$;

(ii) $-z = -(x+iy) = (-x) + i(-y)$.

F5) (i) $1 = 1 + 0i$;

(ii) $z = x + iy \neq 0$;

$$z^{-1} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}. \quad (*)$$

check that $(*)$ is indeed the multiplicative inverse of ~~z=~~ $x+iy$.

Since $\mathbb{C}$ is a field, there holds:

$z_1z_2 = 0 \Rightarrow z_1 = 0$ or $z_2 = 0$ or both,

called <u>null-factor</u> or <u>cancellation law</u>.