Math 3401/3901 Lecture 2

Further motivation:

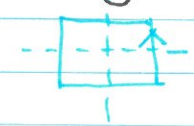
* Evaluating series

E.g. ① $\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}$ (Euler)

② $\frac{1}{1}-\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{12}$ (")

③ $\sum_{k=1}^{\infty} \frac{1}{1+(2\pi k)^2} = \frac{1}{2}\left(\frac{1}{e-1}-\frac{1}{2}\right)$ (?)

① $\frac{1}{2\pi i}\int \frac{\cot \pi z}{z^2}\,dz$



* If you want to buy a house:

Consider the Riemann hypothesis

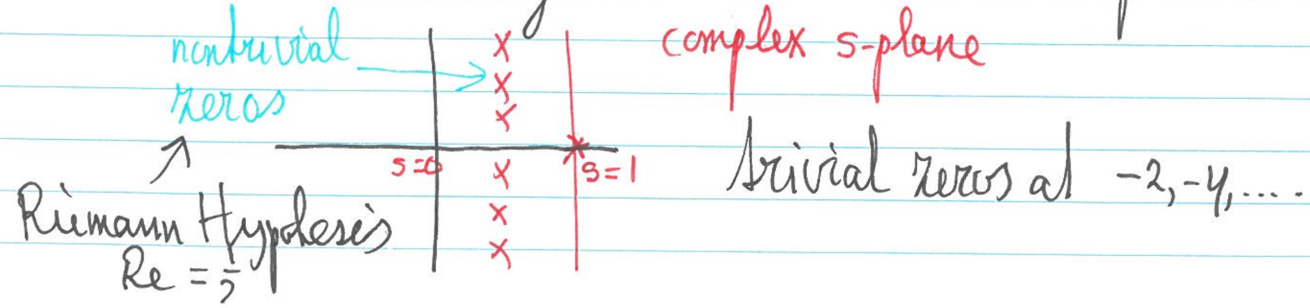
$\zeta : \mathbb{C}\setminus\{1\} \to \mathbb{C}$

defined for $\mathrm{Re}(s)>1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Euler: $\zeta(2) = \frac{\pi^2}{6}$

We need analytic continuation to a fcn on $\mathbb{C}\setminus\{1\}$



Complex numbers

* 1535 Tartaglia: real roots of x^3+ax+b, $a,b>0$
* 1545 (Ars Magna): Cardano, encountered expressions such as $5+\sqrt{-15}$
  "Sophistic" "mental torture" "subtle but useless".
* Decarte 1630~1640 "imaginary numbers"
* Wessel 1799: complex numbers

What we (should) know

* $\mathbb{N} = \{1,2,3,\dots\}$ natural #s
  $\cap$
* $\mathbb{N}_0 = \{0,1,2,\dots\}$ " with 0.
  $\cap$
* $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$ ring of integers: addition & multiplication
  integral domain (commutative, no zero divisors: $xy=0 \Rightarrow x=0$ or $y=0$.
  $\cap$
* $\mathbb{Q} = \{p/q,\ p,q \in \mathbb{Z},\ q \neq 0\} = \mathrm{Frac}(\mathbb{Z})$
  rational #s or field of fractions of $\mathbb{Z}$.
  $\mathbb{Q}$ is countable, but not "closed"
  $\cap$
* $\mathbb{R}$: field of real #'s, uncountable, not algebraically closed
  $\cap$
* $\mathbb{C}$ field of complex #'s, ", algebraically closed

Definition of $\mathbb{C}$

$$\mathbb{C} \underset{\substack{\uparrow \\ \text{as a set}}}{=} \{ (x,y) : x, y \in \mathbb{R} \}$$

We turn this into a field by defining $+$, $\times$ as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) * (x_2, y_2) = \cancel{(x_1 y_2 \ldots)}$$

$$= (x_1 x_2 - y_1 y_2, \; x_1 y_2 + y_1 x_2)$$

<u>Homework</u> Check all field axioms are satisfied

Choose a 'basis' $\underset{\in \mathbb{C}}{1} = (\underset{\in \mathbb{R}}{1}, 0)$, $i = (0, \underset{\in \mathbb{R}}{1})$

$\mathbb{C} = \text{Span}_{\mathbb{R}} \{1, i\}$, i.e.,

$$(x,y) = x \cdot (1,0) + y\,(0,1) = x \cdot 1 + iy = x + iy$$

$$+: \; (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$\times: \; (x_1 + iy_1) \times (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

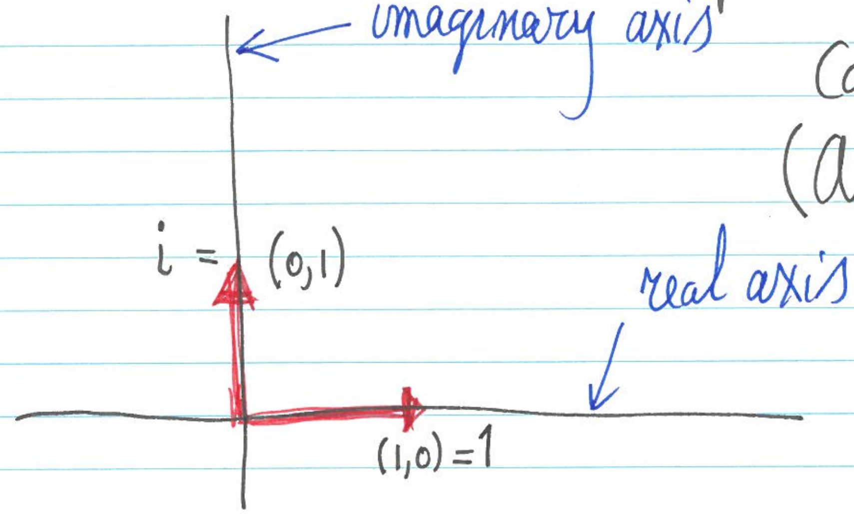
<u>Note</u>: $\boxed{i \times i = -1}$

$\uparrow$

set $x_1 = x_2 = 0$

$y_1 = y_2 = 1$

We represent $\mathbb{C}$ in the 2-dim plane as follows



Complex plane
(Argand plane)

Note: $\mathbb{R}$ is contained in $\mathbb{C}$ by identifying $\overset{\in \mathbb{R}.}{x} + i\,0$ with the real number x

Note: If $z = x+iy$, $(x,y) = (0,0)$ then

$$z^{-1} = \frac{1}{x+iy} = \frac{(x-iy)}{(x-iy)} \cdot \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{\bar{z}}{|z|^2}$$

Friday?