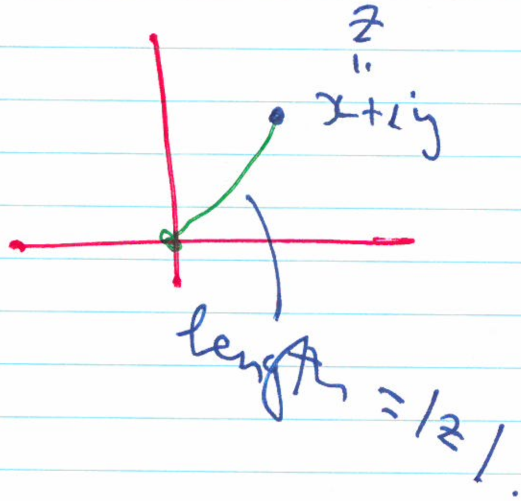
BC §4, 5 other f^n's

* modulus : $|z| = \sqrt{x^2+y^2}$

e.g., $|7+3i| = \sqrt{7^2+3^2} = \sqrt{58}$

$|\cdot| : \mathbb{C} \to \mathbb{R}$

(indeed $|\cdot| : \mathbb{C} \to [0,\infty)$)



* $Re(z)$ = real part of z

$Im(z)$ = imaginary " "

$Re : \mathbb{C} \to \mathbb{R}$

$Im : \mathbb{C} \to \mathbb{R}$.

e.g., $Re(7+3i) = 7$;

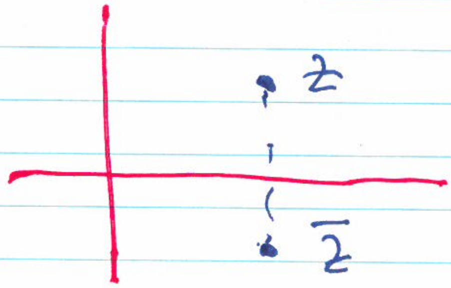
$Im(7+3i) = 3$

not $3i$.

Complex conjugate: $\bar{\cdot} : \mathbb{C} \to \mathbb{C}$

$$x+iy \mapsto x-iy$$

If $z = x+iy$, then $\bar{z} = x-iy$.



geometrically: reflect in the real axis.

Properties:

(i) $z = \bar{z} \iff \mathrm{Im}(z) = 0$, i.e. $z \in \mathbb{R}$;

(ii) $\overline{(\bar{z})} = z$;

(iii) $\overline{(zw)} = \bar{z}\,\bar{w}$;

(iv) $\overline{\left(1/z\right)} = 1/\bar{z}$;

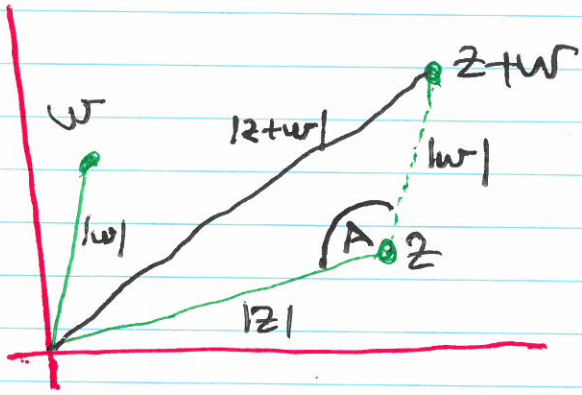
(v) $|z|^2 = z\bar{z}$;

(vi) $\mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$; $\mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$

(vii) $\overline{z+w} = \bar{z} + \bar{w}$.

Very useful: triangle inequality

$z, w \in \mathbb{C} \qquad |z+w| \leq |z| + |w|$



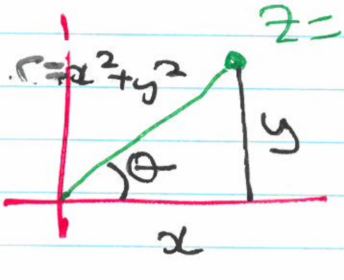
Cos rule $\Rightarrow$

$$|z+w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos A.$$
$$\leq |z|^2 + |w|^2 + 2|z||w|$$
$$= (|z| + |w|)^2$$

Since $\cos A \geq -1$.

Take $\sqrt{\ }$ $\Rightarrow$ done. $\square$

B-C §6-9 Polar coords



$z = x + iy \neq 0$

$x = r\cos\theta$

$y = r\sin\theta$

Write: $z = re^{i\theta}$
$= r\cos\theta + i\sin\theta$
$= r\,\mathrm{cis}\,\theta$

<u>rmk:</u> (*) follows formally from Taylor series for e^x, $\cos x$, $\sin x$.

Here, θ is <u>an argument</u> of z: write $\theta = \arg z$. Here, θ is not a (single-valued) function: if θ is an argument of z, then so is $\theta + 2n\pi$ for any $n \in \mathbb{Z}$.

We (often) want a "unique argument"

<u>Arg(z)</u> is defined to be the unique value of $\arg z$ with $-\pi < \operatorname{Arg} z \leq \pi$.

$$\operatorname{Arg}(1+i) = \pi/4 \;;$$

$$\arg(1+i) = \ldots, -7\pi/4, \pi/4, 9\pi/4, \ldots.$$

$$\operatorname{Arg}(-6) = \pi$$

$\operatorname{Arg}(0)$ is undefined.

"$\arg(0) = \mathbb{R}$"

So, Arg is a f^n: $\underbrace{\mathbb{C} \setminus \{0\}}_{\text{a.k.a } \mathbb{C}^* \text{ or } \mathbb{C}_*}$

set minus

(*) means check!

Notes: * $|e^{i\theta}| = 1$

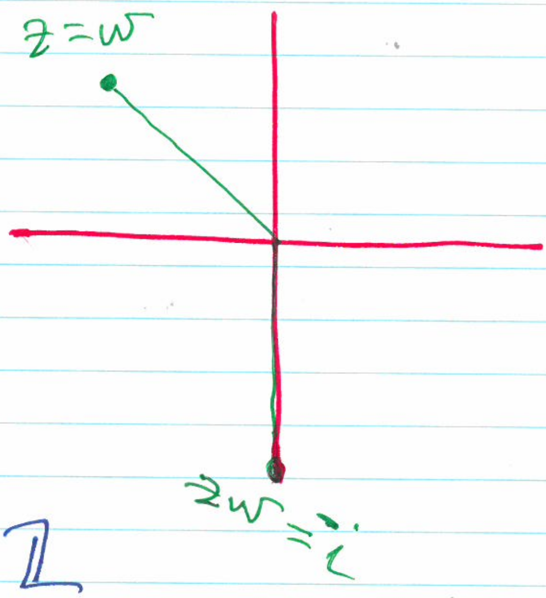
* $(e^{i\theta})^{-1} = e^{-i\theta} = \overline{e^{i\theta}}$

* $(re^{i\theta})(\rho e^{i\phi}) = (r\rho)e^{i(\theta+\phi)}$

$$\left\{\begin{array}{l} |zw| = |z|\cdot|w| \\ \arg(zw) = \arg z + \arg w \end{array}\right.$$

However, $\operatorname{Arg}(zw)$ may fail to be equal to $\operatorname{Arg}(z) + \operatorname{Arg}(w)$.

E.g., $z = w = \dfrac{-1+i}{\sqrt{2}}$.



$\operatorname{Arg}(z) = \operatorname{Arg}(w) = 3\pi/4$

$zw = -i \Rightarrow \operatorname{Arg}(zw) = -\pi/2$

But $\operatorname{Arg}(z) + \operatorname{Arg}(w) = 3\pi/2$.

* $z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta} \quad n \in \mathbb{Z}$

$r = 1$: $e^{in\theta} = (\cos\theta + i\sin\theta)^n$

$= \cos(n\theta) + i\sin(n\theta)$ de Moivre

e.g. $(1+i)^7 = (\sqrt{2}e^{i\pi/4})^7$

$= (\sqrt{2})^7 e^{7\pi i/4} = 8 - 8i$.

Roots of a complex # (BC §10)

Find the nth roots of $z = re^{i\theta}$ $\in \mathbb{C}_*$, $n > 1$, $n \in \mathbb{Z}$

means: find all $w \in \mathbb{C}$ s.t. $w^n = z$.

Notation: $\exp(\xi) = e^{\xi}$
xi

z has n distinct nt roots:

$$\left\{ r^{1/n}\exp\left(\frac{i\theta}{n}\right), r^{1/n}\exp\left(\frac{i\theta}{n} + \frac{i2\pi}{n}\right), r^{1/n}\exp\left(\frac{i\theta}{n} + \frac{i4\pi}{n}\right), \ldots r^{1/n}\exp\left(\frac{i\theta}{n} + \frac{i2(n-1)\pi}{n}\right) \right\}.$$

example: tutes.