LECTURE 3

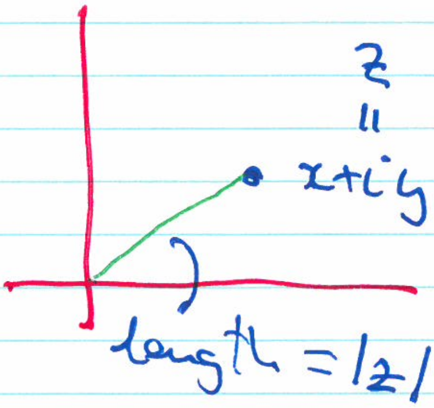
BC §4, 5 — other f^n's.

* modulus : $z = x+iy$, $|z| = \sqrt{x^2+y^2}$

e.g. $|7+3i| = \sqrt{7^2+3^2} = \sqrt{58}$

$|\cdot| : \mathbb{C} \to \mathbb{R}$

(indeed, $|\cdot| : \mathbb{C} \to [0,\infty)$).



* $\mathrm{Re}(z)$ = real part of z

$\mathrm{Im}(z)$ = imaginary " "

$\mathrm{Re} : \mathbb{C} \to \mathbb{R}$

$\mathrm{Im} : \mathbb{C} \to \mathbb{R}$.

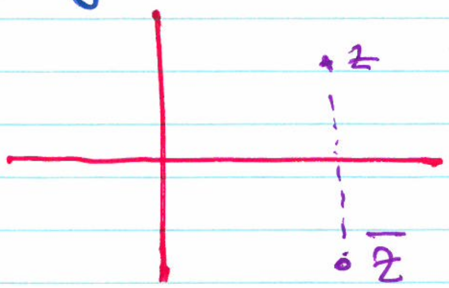
e.g. $\mathrm{Re}(7+3i) = 7$;

$\mathrm{Im}(7+3i) = 3$ — not $3i$.

Complex conjugate $\bar{\cdot} : \mathbb{C} \to \mathbb{C}$

$x+iy \mapsto x-iy$

If $z = x+iy$ $\bar{z} = x-iy$.



geometrically:
reflect in the real axis.

Properties:

(i) $z = \bar{z} \iff \mathrm{Im}(z) = 0$, i.e., $z \in \mathbb{R}$;

(ii) $\bar{\bar{z}} = z$;

(iii) $\overline{zw} = \bar{z}\,\bar{w}$;

(iv) $\overline{\left(1/z\right)} = 1/\bar{z}$

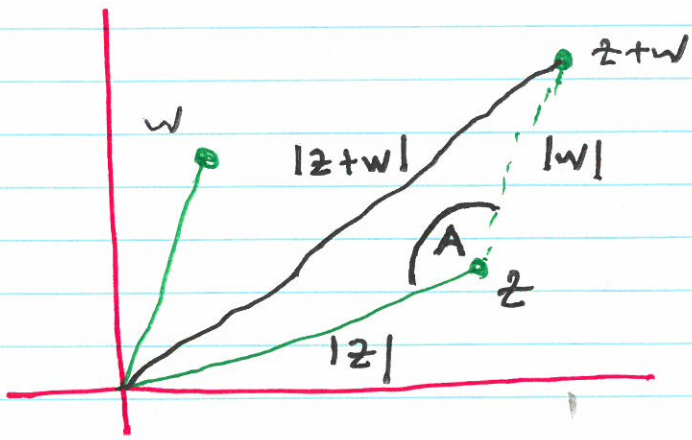
(v) $|z|^2 = z\bar{z}$

(vi) $\mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$; $\mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$

(vii) $\overline{z+w} = \bar{z} + \bar{w}$.

Very useful : triangle inequality.

For $z, w \in \mathbb{C}$, $|z+w| \leq |z| + |w|$.



cos rule :

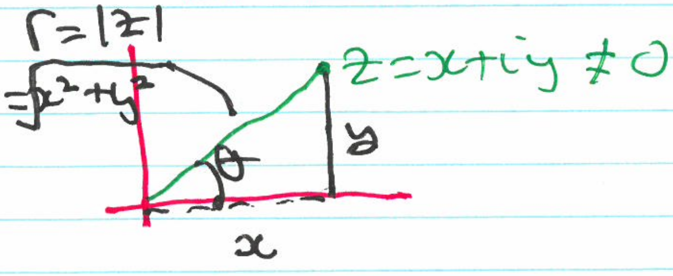
$$|z+w|^2 = |z|^2 + |w|^2 - 2|z||w| \cos A$$

$$\leq |z|^2 + |w|^2 + 2|z| \cdot |w| \quad \text{(since } \cos A \geq -1\text{)}$$

$$= (|z| + |w|)^2$$

Take $\sqrt{\ }$ $\Rightarrow$ done $\square$

B-C §6-9 Polar coords.



$x = r\cos\theta$

$y = r\sin\theta$

Write : $z = re^{i\theta}$

$= r(\cos\theta + i\sin\theta)$ ✱

$= r \operatorname{cis}\theta$

<u>rmk</u>: * follows formally from Taylor series for e^x, $\cos x$, $\sin x$

Here, θ is <u>an argument</u> of z: write $\theta = \arg z$. Here θ is not a (single-valued) function: if θ is an argument of z, then so is $\theta + 2n\pi$ for any $n \in \mathbb{Z}$.

We often want a "unique argument".

<u>Arg(z)</u> is defined to be the unique value of $\arg(z)$ with $-\pi < \operatorname{Arg} z \leq \pi$

e.g. $\operatorname{Arg}(1+i) = \pi/4$;

$\arg(1+i) = \ldots, -7\pi/4, \pi/4, 9\pi/4, \ldots$

$\operatorname{Arg}(-6) = \pi$

$\operatorname{Arg}(0)$ is undefined.

in some sense, "$\arg(0) = \mathbb{R}$"

So Arg is a f^n $\underbrace{\mathbb{C} \setminus \{0\}}_{\text{a.k.a. } \mathbb{C}^* \text{ or } \mathbb{C}_*}$ (\set minus)

Notes: $|e^{i\theta}| = 1$ $\quad$ $\star$ means check!

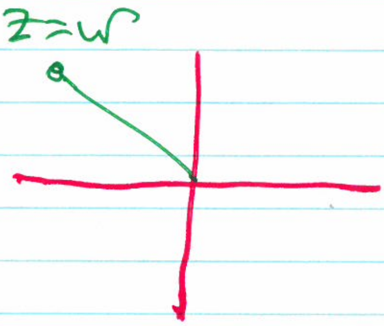
$$(e^{i\theta})^{-1} = e^{-i\theta} = \overline{e^{i\theta}}$$

$$(re^{i\theta})(\rho e^{i\phi}) = (r\rho)e^{i(\theta+\phi)}$$

$$|zw| = |z|\cdot|w|$$

$$\arg(zw) = \arg z + \arg w$$

However, $\operatorname{Arg} zw$ may fail to be equal to $\operatorname{Arg}(z) + \operatorname{Arg}(w)$.



E.g. $z = w = \frac{-1+i}{\sqrt{2}}$

$\operatorname{Arg}(z) = \operatorname{Arg}(w) = 3\pi/4$.

$zw = -i \Rightarrow \operatorname{Arg}(zw) = -\pi/2$

But $\operatorname{Arg} z + \operatorname{Arg} w = 3\pi/2$.

$\star$ $z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$ $\quad n \in \mathbb{Z}$

$r=1$: $e^{in\theta} = (\cos\theta + i\sin\theta)^n$

$= \cos(n\theta) + i\sin(n\theta)$ $\quad$ de Moivre

e.g. $(1+i)^7 = (\sqrt{2}e^{i\pi/4})^7$

$= (\sqrt{2})^7 e^{7\pi i/4} = 8 - 8i$

Roots of a complex # (BC §10)

Find the nth roots of $z = re^{i\theta}$, $n>1$, $n\in\mathbb{Z}$
$z \in \mathbb{C}_*$

means: find all $w \in \mathbb{C}$ s.t. $w^n = z$.

Notation: $\exp(\zeta) = e^{\zeta}$

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z has n distinct roots:

$$\left\{ r^{1/n}\exp\left(\frac{i\theta}{n}\right),\ r^{1/n}\exp\left(\frac{i\theta}{n}+\frac{i2\pi}{n}\right),\ r^{1/n}\exp\left(\frac{i\theta}{n}+\frac{i4\pi}{n}\right),\ \ldots\ r^{1/n}\exp\left(\frac{i\theta}{n}+\frac{i2(n-1)\pi}{n}\right)\right\}$$

example: pracs.