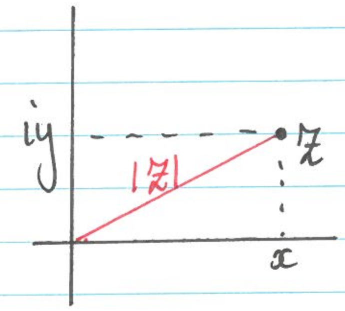
Math 3401/3901 Lecture 3

From last time: $z = x+iy$, $(x,y) \neq (0,0) \Rightarrow$

$$z^{-1} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2}$$

* modulus of a complex number $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$

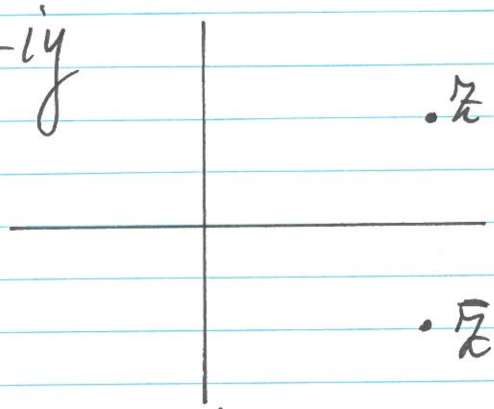
$$z = x+iy \mapsto \sqrt{x^2+y^2}$$



* complex conjugate : $\bar{\cdot} : \mathbb{C} \to \mathbb{C}$

$$z = x+iy \mapsto \bar{z} = x-iy$$

"reflection in the real axis"



This map is an example of an involution, i.e.,

$$\bar{\bar{z}} = z$$

* $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$, $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$

$z = x+iy$, $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$ (not iy !)

Eg $\mathrm{Re}(3+6i) = 3$, $\mathrm{Im}(3+6i) = 6$.

Properties (Homework)

(i) $z = \bar{z} \iff \operatorname{Im}(z) = 0$, i.e., $z \in \mathbb{R}$

(ii) $\bar{\bar{z}} = z$ (conjugation is an involution)

(iii) $\overline{zw} = \bar{z} \cdot \bar{w}$ & $\overline{z+w} = \bar{z} + \bar{w}$

(iv) $\overline{z^{-1}} = \frac{1}{\bar{z}}$ $(z \neq 0)$

(v) $|z|^2 = z\bar{z}$

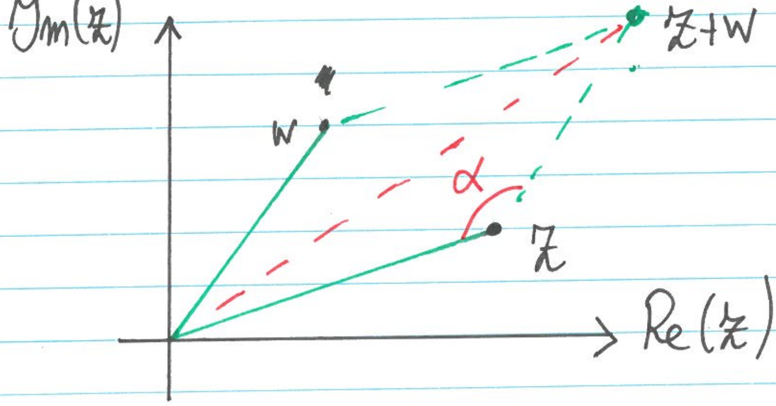
(vi) $\operatorname{Re}(z) = \frac{z+\bar{z}}{2}$, $\operatorname{Im}(z) = \frac{z-\bar{z}}{2i}$

etc.

Lemma (triangle inequality)

For $z, w \in \mathbb{C}$, $|z+w| \leq |z| + |w|$.

Pf



$$|z+w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\alpha$$

$$\leq |z|^2 + |w|^2 + 2|z||w|$$

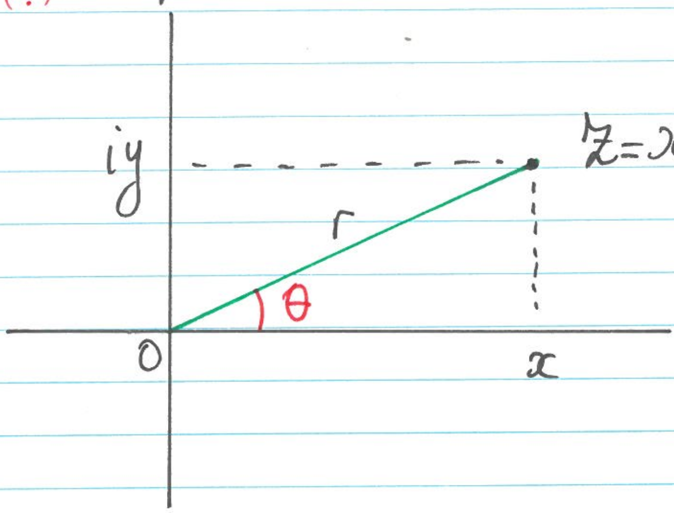
$\cos\alpha \geq -1$

$$= (|z| + |w|)^2$$

Take $\sqrt{\ }$ and we are done $\square$

Polar coordinates

Instead of representing a complex number using ~~"Euclidean"~~ "Cartesian" coordinates, we can use "the" angle and modulus of a complex #.
(?)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow z = r(\cos\theta + i\sin\theta)$$

$|r| = |z| = \sqrt{x^2+y^2}$

"$\theta = \arctan\left(\frac{y}{x}\right)$"

Homework: Check that if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ & $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $\cancel{z_1 z_2 = z_1}$

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \right)$$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ (currently this is somewhat justified)

$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

We will write $\theta = \arg(z)$, but $\arg: \mathbb{C}^* \to \mathbb{R}$ is not a proper function, but is a multivalued function: If $\arg(z) = \theta$, then so is $\theta + 2\pi k$, $k \in \mathbb{Z}$. Sometimes we want uniqueness, in which case we fix a value (fix a branch). The most common choice is $-\pi < \theta \leq \pi$, in which case we write $\operatorname{Arg}(z)$:

$$\operatorname{Arg}: \mathbb{C}^* \to (-\pi, \pi].$$

$\mathbb{C} \setminus \{0\}$: There is no argument / Argument attached to the origin.

<u>Ex</u> $\operatorname{Arg}(1+i) = \dfrac{\pi}{4}$

$\arg(1+i) = \dfrac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$.

$\operatorname{Arg}(0)$ & $\arg(0)$ are undefined.

Nice properties of $e^{i\theta}$ & arg/Arg

* $|e^{i\theta}| = 1$
* $(e^{i\theta})^{-1} = e^{-i\theta}$
* $e^{i\theta} \cdot e^{i\varphi} = e^{i(\theta+\varphi)}$ (as before)
* $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
* $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

* $\arg(zw) = \arg(z) + \arg(w)$
* $\operatorname{Arg}(zw) \neq \operatorname{Arg}(z) + \operatorname{Arg}(w)$

in general

$$\operatorname{Arg}((1+i)\cdot(1+i)) = \operatorname{Arg}(2i) = \frac{\pi}{2}$$
$$\operatorname{Arg}(1+i) + \operatorname{Arg}(1+i) = \frac{\pi}{4} + \frac{\pi}{4}$$

$$\operatorname{Arg}((i-1)\cdot(i-1)) \neq \operatorname{Arg}(i-1) + \operatorname{Arg}(i-1).$$