LECTURE 4

B-C §13 (8Ed §12,13)

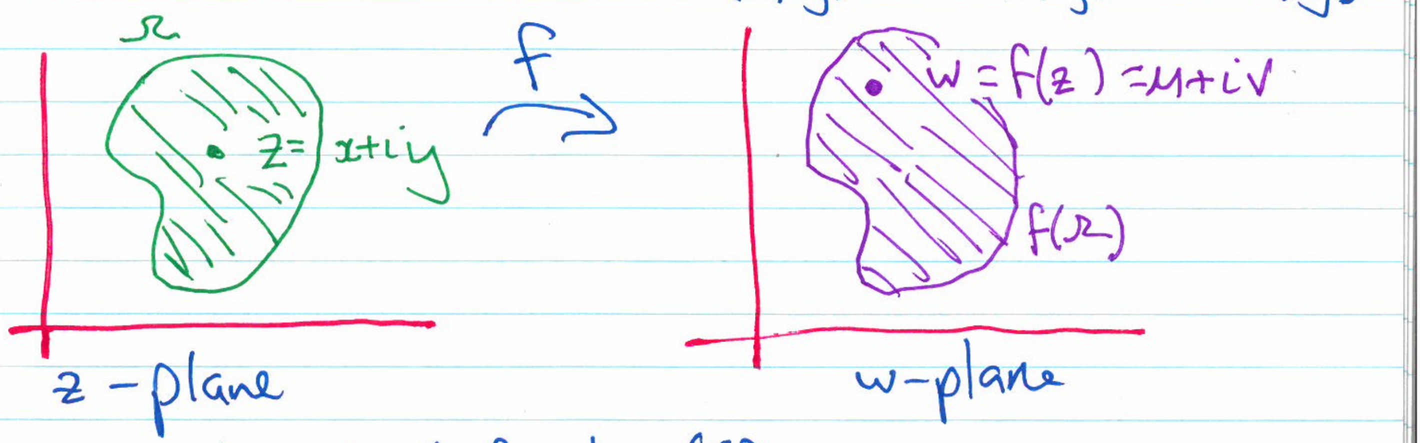
Functions & Mappings.

$\Omega \subseteq \mathbb{C}$: A function $f : \Omega \to \mathbb{C}$ can be viewed as a mapping on Ω, the domain of f.
If Ω is not specified, take it as large as possible.

E.g. for $f(z) = 1/z$, domain is $\mathbb{C}_*$ $(= \mathbb{C} \setminus \{0\})$
& $f : \mathbb{C}_* \to \mathbb{C}$ (indeed, $f : \mathbb{C}_* \to \mathbb{C}_*$).

We can write $f : z \mapsto 1/z$; or $w = 1/z$;
mapsto
or just $1/z$.

Usual notation: $w \quad (x,y) \to (u,v)$
$(x,y,u,v \in \mathbb{R})$ i.e., $w(x+iy) = u(x+iy) + iv(x+iy)$
or
$w(x,y) = u(x,y) + iv(x,y)$



* Ω = domain of f = dom(f) ,
* Range(f) = $f(\Omega)$ = $\{w : w = f(z)$ for some $z \in \Omega\}$;
* $f^{-1}(\xi) = \{z \in \Omega : f(z) = \xi\}$

f^{-1} is the inverse of f (not $1/f$). It is not necessarily a function!

Examples:

* $f(z) = 1/z$: $\mathrm{dom}(f) = \mathbb{C}_*$, $f^{-1}(\xi) = 1/\xi$ is a function, $\mathbb{C}_* \to \mathbb{C}_*$.

* $g(z) = 1/(1-|z|^2)$. $\mathrm{Dom}(g) = \{z : |z| \neq 1\}$,
  $g : \mathrm{dom}(g) \to \mathbb{R}$.
  Inverse of g is not a f^n: if $g(z) = \rho$, then $g(e^{i\theta} z) = \rho$ for any $\theta \in \mathbb{R}$ ☆.

* E.g. $h(z) = z^2$: $h : \mathbb{C} \to \mathbb{C}$ inverse is not a function.

Aim: get a geometric idea of what a given function does.

E.g. $w = 1 + z$ moves every point one (unit) to the right;

E.g. $re^{i\theta} \mapsto re^{i(\theta + \pi/2)}$ rotates through an angle of $\pi/2$ in the positive (counter-clockwise) direction about the origin.

Basic idea with new mappings: break it down into compositions of known/easy maps.

Ex 1 : Linear transformations

$w = Az + b \qquad A, b \in \mathbb{C}, \; A \neq 0.$

Note: domain is $\mathbb{C}$.

split up into: (i) dilation & rotation;
(ii) translation.

(i) $z \mapsto Az$ write $A = ae^{i\alpha}$, $\alpha \in \mathbb{R}, a \in \mathbb{R}_+$
($a = |A|$, α is an argument of A)

$re^{i\theta} \mapsto ae^{i\alpha} re^{i\theta} = (ar)e^{i(\theta+\alpha)}$

Geometrically: dilates (expands, contracts or leaves the same) modulus by a factor of $a = |A|$, & rotates through an angle of $\alpha = \arg A$ in the positive direction (any argument of A, as they all differ by integer multiples of 2π).

(ii) $z \mapsto z + b \qquad b = b_1 + b_2 i$

b_1 & $b_2 \in \mathbb{R}$

$\Rightarrow$ translate b_1 to the right & b_2 up.

$z \mapsto Az + b$: compose, i.e., do (i), then (ii).

Next: $z \mapsto 1/z$.