LECTURE 4

Rmk Asst 1 q5.

The ordering referred to is ordering as a field.

$\mathbb{C}$ can be totally ordered in the set theoretic sense, i.e., you can find a binary relation $\preceq$ on $\mathbb{C}$ s.t for $z, w, \zeta \in \mathbb{C}$, then:

① $z \preceq z$

② $z \preceq w$ & $w \preceq \zeta \Rightarrow z \preceq \zeta$

③ $z \preceq w$ & $w \preceq z \Rightarrow z = w$

④ $\forall z, w \in \mathbb{C}$, $z \preceq w$ or $w \preceq z$

$z \prec w \Leftrightarrow z \preceq$ & $z \neq w$.

E.g. lexicographical ordering:

$$x_1 + iy_1 \leq x_2 + iy_2 \Leftrightarrow \begin{cases} x_1 \leq x_2 & \text{if } x_1 \neq x_2 \\ y_1 \leq y_2 & \text{if } x_1 = x_2 \end{cases}$$

Check: $\preceq$ is a total ordering on $\mathbb{C}$.

Not compatible with the field structure, i.e., not an ordering in the sense of Asst.1 q5

$1 - i > 0$ but $(1-i)\cdot(1-i) < 0$.

B-C §13 (8Ed §12,13)

Functions & Mappings

$\Omega \subseteq \mathbb{C}$: A f^n $f : \Omega \to \mathbb{C}$ can be viewed as a mapping on Ω, the domain of f.

If Ω is not specified, take it as large as possible.

E.g. For $f(z) = 1/z$, domain is $\mathbb{C}_*$ $(= \mathbb{C} \setminus \{0\})$ & $f : \mathbb{C}_* \to \mathbb{C}$ (indeed, $f : \mathbb{C}_* \to \mathbb{C}_*$)

We can write $f : z \mapsto 1/z$; or $w = 1/z$;

\mapsto

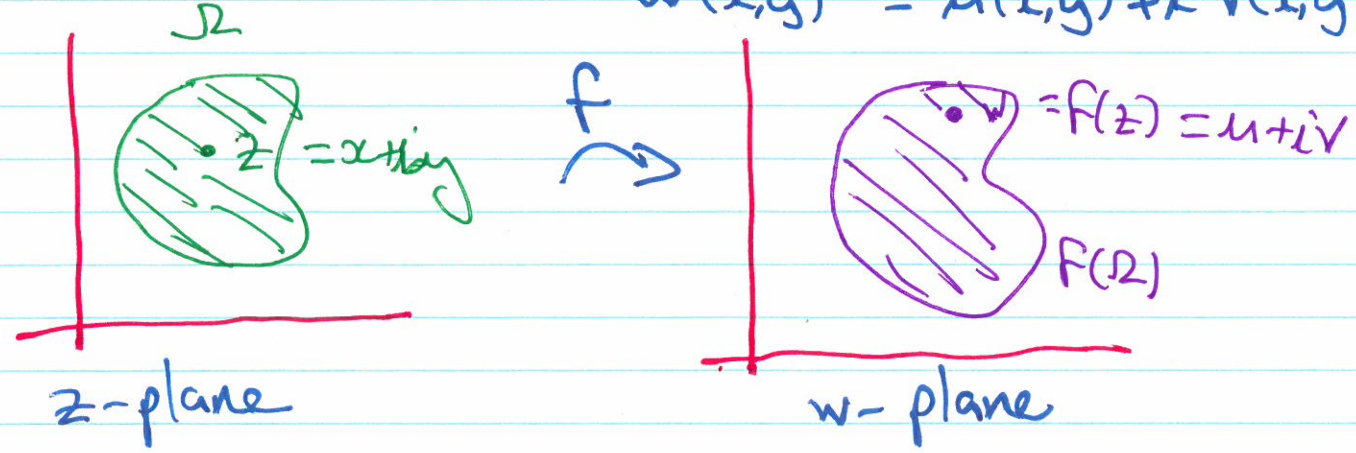
or just $1/z$.

Usual notation: $w : (x,y) \to (u,v)$

$(x,y,u,v \in \mathbb{R})$ i.e., $w(x+iy) = u(x+iy) + i\,v(x+iy)$

or

$w(x,y) = u(x,y) + i\,v(x,y)$



* Ω = domain of f = dom(f)
* Range(f) = $f(\Omega) = \{w : w = f(z) \text{ for some } z \in \Omega\}$;
* $f^{-1}(\zeta) = \{z \in \Omega : f(z) = \zeta\}$.

f^{-1} is the inverse of f (different from $1/f$!)
It is not necessarily a function.

Examples:

* $f(z) = 1/z$: $\mathrm{dom}(f) = \mathbb{C}_*$, $f^{-1}(\zeta) = 1/\zeta$ is a function, $\mathbb{C}_* \to \mathbb{C}_*$.
* $g(z) = 1/{1-|z|^2}$. $\mathrm{Dom}(g) = \{z : |z| \neq 1\}$.
  $g : \mathrm{dom}(g) \to \mathbb{R}$.
  Inverse of g is not a f^n: if $g(z) = \rho$, then $g(e^{i\theta} z) = \rho$ for any $\theta \in \mathbb{R}$ ☆.
  E.g. $h(z) = z^2$: $h : \mathbb{C} \to \mathbb{C}$, inverse is not a f^n.

Aim: get a geometric idea of what a given f^n does.

E.g. $w = 1 + z$ moves every point one (unit) to the right;

E.g. $re^{i\theta} \mapsto re^{i(\theta + \pi/2)}$ rotates through an angle of $\pi/2$ in the positive (counter-clockwise) direction about the origin.

Basic idea with new mappings: break them down into compositions of known/easy maps.

Ex 1 Linear transformations.

$w = Az + b \quad A, b \in \mathbb{C} \quad A \neq 0$.

Note: domain is $\mathbb{C}$.

Split up into (i) dilation & rotation;
(ii) translation.

(i) $z \mapsto Az$: write $A = ae^{i\alpha}$ $\alpha \in \mathbb{R}$, $a \in \mathbb{R}_+$
($a = |A|$, α is an argument of A)

$re^{i\theta} \mapsto ae^{i\alpha} re^{i\theta} = (ar)e^{i(\theta+\alpha)}$

Geometrically: dilates (expands, contracts or leaves the same) modulus by a factor of $a = |A|$, & rotates through an angle of $\alpha = \arg A$ in the positive sense/direction (any argument of A, as they differ by multiples of 2π.

(ii) $z \mapsto z + b \quad b = b_1 + b_2 i$

b_1 & $b_2 \in \mathbb{R}$

$\Rightarrow$ translate b_1 to the right & b_2 up.

$z \mapsto Az + b$: compose, i.e., do (i), then do (ii)

Next : $z \mapsto 1/z$.