Math 3401/3901 Lecture 4

Recall our list of nice properties of polar coordinates.

Some more niceness:

* $$e^{i(n\theta)} \overset{①}{=} (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

$$\overset{②}{=} \cos n\theta + i\sin n\theta$$

de Moivre: $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

Example: $n=2$

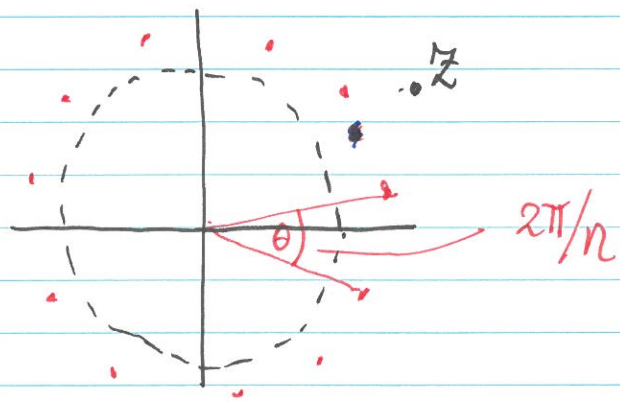
$$(\cos\theta + i\sin\theta)^2 = \boxed{(\cos^2\theta - \sin^2\theta)} + 2i\,\boxed{\sin\theta\cos\theta}$$

$$= \boxed{\cos 2\theta} + i\,\boxed{\sin 2\theta}$$

* Helps with finding nth roots:

Solve $z = w^n$, $z = re^{i\theta}$

$$w = r^{1/n} e^{i(\theta/n) + 2\pi k i/n}, \quad k = 0, 1, \dots, n-1$$



Homework: show these are all of the solutions.

Function on $\mathbb{C}$ (or subsets thereof)

$\Omega \subseteq \mathbb{C}$. We wish to study fcn^s $f : \Omega \to \mathbb{C}$.

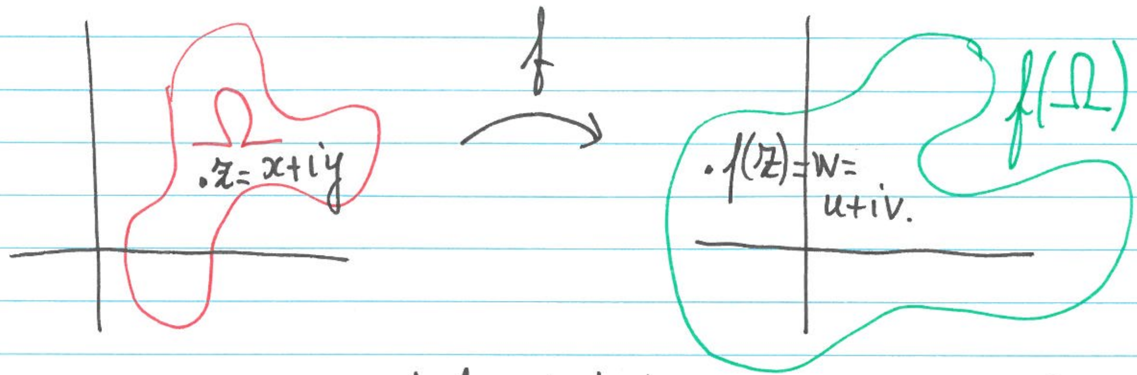
Examples

$f : \mathbb{C}^* \to \mathbb{C}, \quad z \mapsto 1/z$

(or $f : \mathbb{C}^* \to \mathbb{C}^*$)

We often write $w = f(z) = u(z) + i\,v(z)$, where $z = x+iy$, $u(z) = u(x+iy)$, $i\,v(z) = i\,v(x+iy)$

$(x, y) \mapsto (u, v)$



* Ω : domain of f (if Joe does not give Ω, then it is assumed to be as large as possible)
* Range$(f) = f(\Omega) \subseteq \mathbb{C}$
  $= \{ w \in \mathbb{C} : w = f(z) \text{ for some } z \in \Omega \}$.
* $f^{-1}(w)$: preimage of w or (fibre of w)

  Warning f^{-1} is not generally a function!

$$f^{-1}(w) = \{z \in \Omega : f(z) = w\}$$

Example

* $f : \mathbb{C}^* \to \mathbb{C}$, $f(z) = 1/z$

$\mathbb{C}^* = \Omega = \text{domain}$

$$f(\mathbb{C}^*) = \mathbb{C}^*$$

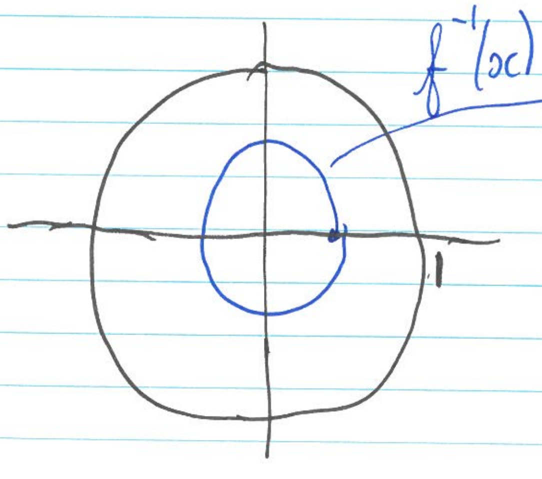
$f^{-1}(z) = z^{-1}$ an actual fcn!

* $f : \underbrace{\mathbb{C} \setminus \{z \in \mathbb{C} : |z| = 1\}}_{\Omega} \to \mathbb{R}$

$$z \mapsto \frac{1}{1 - |z|^2}$$

$f(\Omega)$?

$f(\Omega)$



f^{-1} : not a function!

Aim (typically): Get some geometric idea/interpretation of (what) f is about.

Eg: $w = f(z) = 1 + z$ translation to the right by 1.

$\cancel{w = f(z) = re^{i\theta}}$

$w = f(z) = z e^{i\theta}$, θ fixed ($\in \mathbb{R}$).

rotation in $\mathbb{C}$ by an angle of θ

In general this hard. We try to ~~decompose~~ understand f as a composition of more managable fcn^s.

The simples types of fcn^s we can understand this way are <u>linear transformations</u>:

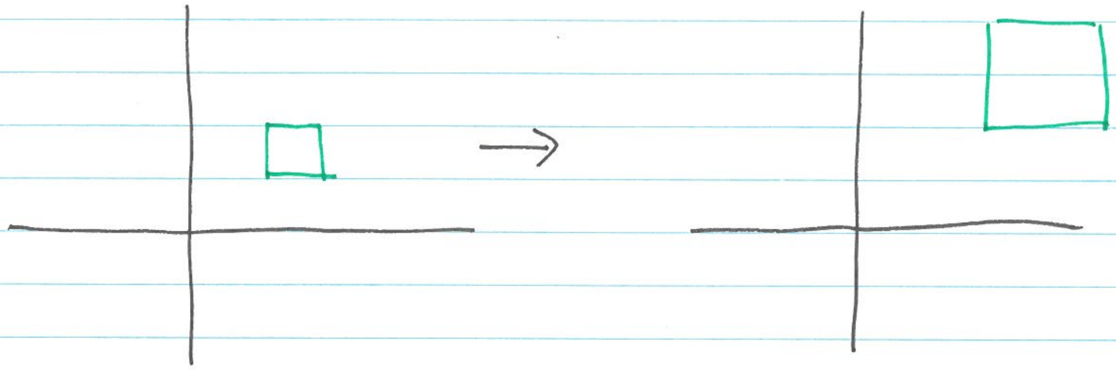
$w = f(z) = Az + b$, $A \neq 0$

Linear transformations are made up of

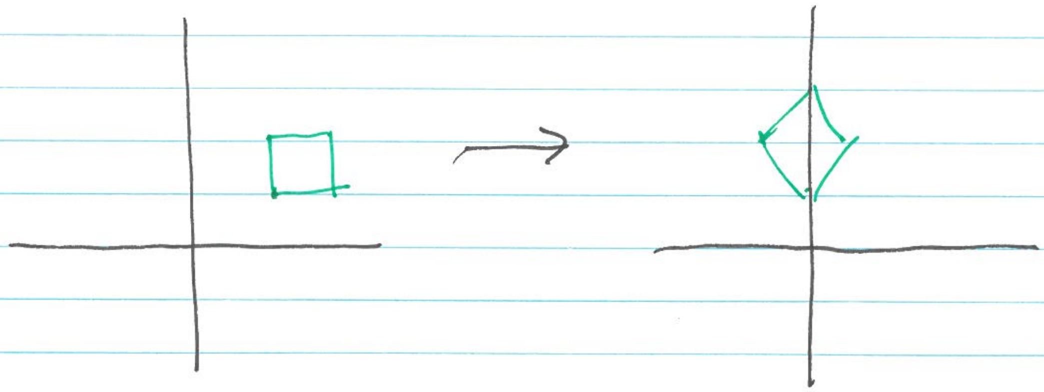
– rotation , – dilation , – translation

~~Eg of~~ Example of a dilation: $f(z) = 2z$

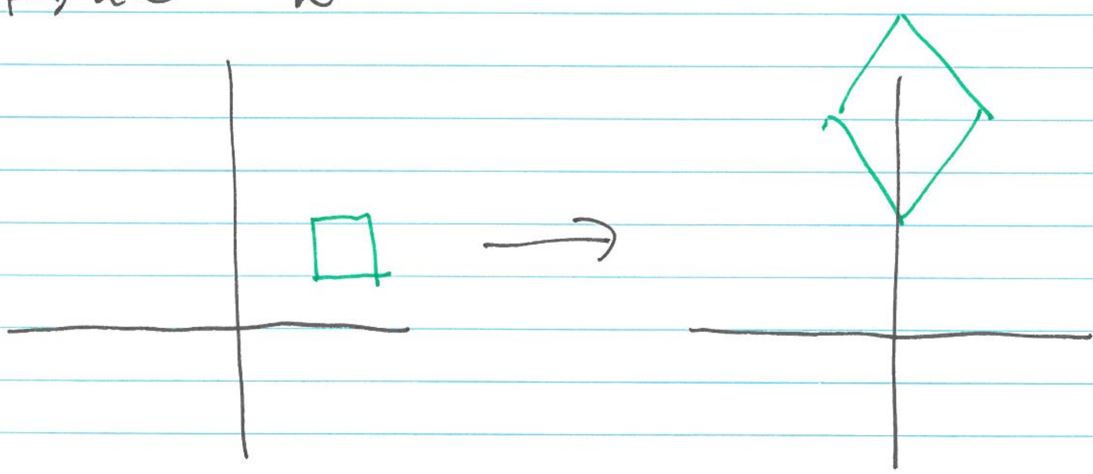
$z \mapsto 2z$



$z \mapsto e^{\pi i/4} z$



$z \mapsto 2e^{\pi i/4} z$



Assignment

You can put a (total) order on $\mathbb{C}$:

$$x_1 + iy_1 > x_2 + iy_2 :$$

if $\quad x_1 > x_2$

or $\quad x_1 = x_2 \wedge y_1 > y_2$

Does not respect the structure of a field. !