LECTURE 5

Ex 2: $z \mapsto 1/z$ on $\mathbb{C}_*$

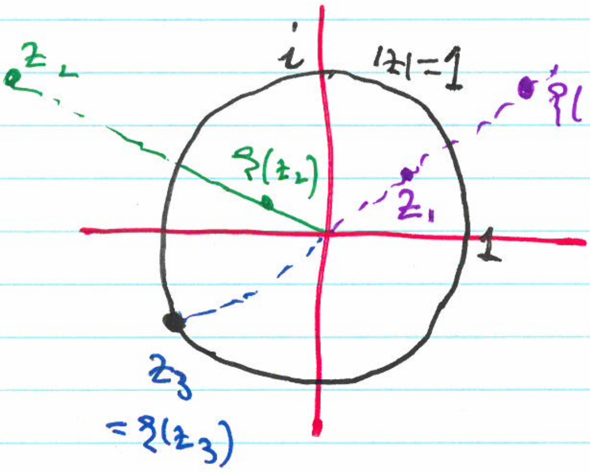
Define $\xi(z) = z/|z|^2$ on $\mathbb{C}_*$;

$\eta(\xi) = \bar{\xi}$ on $\mathbb{C}$.

$$\eta \circ \xi(z) = \eta(\xi(z)) = \overline{\xi(z)} = \overline{\left(z/|z|^2\right)}$$

composed with

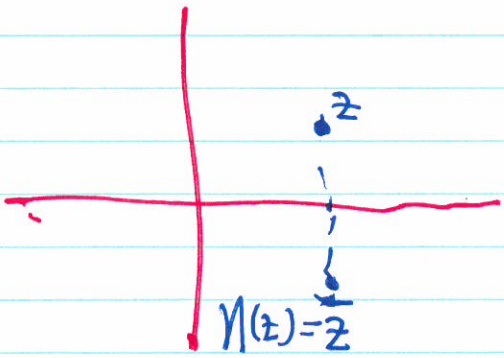
$$= \bar{z}/|z|^2 = \cancel{\bar{z}}/(z\cancel{\bar{z}}) = 1/z .$$



ξ is called inversion (w.r.t. the unit circle)

with respect to.

η is reflection in the real axis.



So, $z \mapsto 1/z$ is inversion, then reflection in the $\mathbb{R}$ axis.

For $w = 1/z = \bar{z}/|z|^2$ with

$x+iy \mapsto u+iv$

$w = \dfrac{x-iy}{x^2+y^2}$, so

$u = \dfrac{x}{x^2+y^2}$ (1) & $v = \dfrac{-y}{x^2+y^2}$ (2)

We can use this to show:

$1/z$ maps circles & lines in the z-plane to circles & lines in the w-plane.

<u>Note:</u> circles & lines in the z-plane have the form

$$\left.\begin{array}{l} A(x^2+y^2) + Bx + Cy + D = 0 \\ A, B, C, D \in \mathbb{R} \quad B^2 + C^2 > 4AD \end{array}\right\} (*)$$

$A = 0$ line ; $A \neq 0$ circle.

From (*), we have

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$$

for $A \neq 0$.

For $w = 1/z$, (1) & (2) $\Rightarrow$

$$D(u^2+v^2) + Bu - Cv + A = 0 \qquad \star$$

Terminology reminders:

* $f: \Omega \to \mathbb{C}$ is 1-1 or injective says:

$\Omega \subseteq \mathbb{C}$ $\quad f(z) = f(\zeta) \Rightarrow z = \zeta$.

* $f: \Omega \to \Lambda \subseteq \mathbb{C}$ is onto or surjective says:

given $\eta \in \Lambda$, $\exists$ (at least one) $z \in \Omega : f(z) = \eta$.

there exists

1-1 & onto : bijective.

Möbius transformations : B-C §99 (8Ed §93ff.)

Def^{n}: Let $a,b,c,d \in \mathbb{C}$ with $\boxed{ad - bc \neq 0}$ (3)

Then $\boxed{w = T(z) = \dfrac{az+b}{cz+d}}$ (4)

is a Möbius (a.k.a linear fractional) transformation.

* $c = 0$: $\mathrm{dom}(w) = \mathbb{C}$ (note $c = 0 \overset{(3)}{\Rightarrow} d \neq 0$)
* $c \neq 0$ $\quad \mathrm{dom}(w) = \mathbb{C} \setminus \{-d/c\}$.

Goal: understand T geometrically.

CASE I $c = 0$

Claim: T maps $\mathbb{C} \to \mathbb{C}$ 1-1 & onto.

Pf: (i) 1-1 Suppose $T(z) = T(\zeta)$ (**)

WTS: $z = \zeta$ (**) $\Rightarrow \frac{a}{d} z + \frac{b}{d} = \frac{a}{d}\zeta + \frac{b}{d}$

want to show

$-b/d$, $\times d/a$ $\Rightarrow z = \zeta$ ∎

(ii) onto: given $w \in \mathbb{C}$, need $z \in \mathbb{C}$ s.t. $T(z) = w$.

Check: $z = \frac{d}{a}(w - b/d)$ works ☆. □

Rmk: $z \mapsto T(z) = \frac{az+b}{d} = \frac{a}{d} z + \frac{b}{d}$

$\frac{a}{d}$: A, $\frac{b}{d}$: "b"

is a linear transformation as previously covered.

CASE II: $c \neq 0$.

$$\underset{T(z)}{w} = \frac{az+b}{cz+d} = \frac{a(z + d/c) - ad/c + b}{c(z + d/c)}$$

$$= a/c + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}.$$

So, in case II, T is the composition of $Z_1 = cz+d$; $W = 1/Z_1$, $w = a/c + \frac{bc-ad}{c} W$.

So in both cases I & II T is a composition of maps previously studied.