Math 3401/3901 Lecture 5

The function $f: \mathbb{C}^* \to \mathbb{C}$ (or $\mathbb{C}^*$)

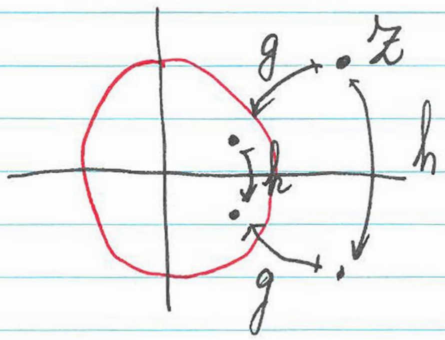
$$z \mapsto \frac{1}{z}$$

$$re^{i\theta} \mapsto \frac{1}{r} e^{-i\theta}$$

$g: \mathbb{C}^* \to \mathbb{C}$ $\qquad re^{i\theta} \mapsto \frac{1}{r} e^{i\theta}$

$h: \mathbb{C} \to \mathbb{C}$ $\qquad \begin{cases} z \mapsto \bar{z} \\ re^{i\theta} \mapsto re^{-i\theta} \end{cases}$

$$f = g \circ h = h \circ g$$



g: 'inversion' wrt the unit circle: $|z| \cdot |g(z)| = 1$

h: complex conjugation (reflection in the real axis)

We want to better understand f globally, not just locally.

$$f(z) = \frac{1}{z} = w = \frac{\bar{z}}{|z|^2}$$

$z = x + iy, \quad w = u + iv$

Solve for u & v: $\boxed{u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}}$ $(*)$

Note: we also have $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$ $(*)'$

Claim: f maps circles & lines to circles & line

Note that circles & lines in the complex plane have the form

$$\boxed{A(x^2+y^2) + Bx + Cy + D = 0} \quad (**)$$

(not all 0) $\Rightarrow A, B, C, D \in \mathbb{R}$, $B^2 + C^2 > 4AD$

$A = 0$: , $(B, C) \neq (0,0)$ line $\quad r^2 \quad \left(r = \frac{\sqrt{B^2+C^2-4AD}}{2|A|}\right)$

$A \neq 0$: $\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$

Sub $(*)'$ into $(**)$

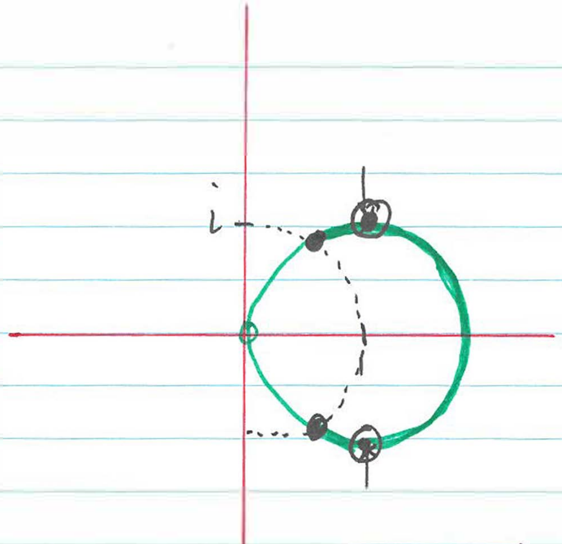
$$A\left(\frac{\cancel{u^2+v^2}}{(u^2+v^2)^{\cancel{2}}}\right) + B\frac{u}{u^2+v^2} + C\frac{-v}{u^2+v^2} + D = 0 \quad D \neq 0$$

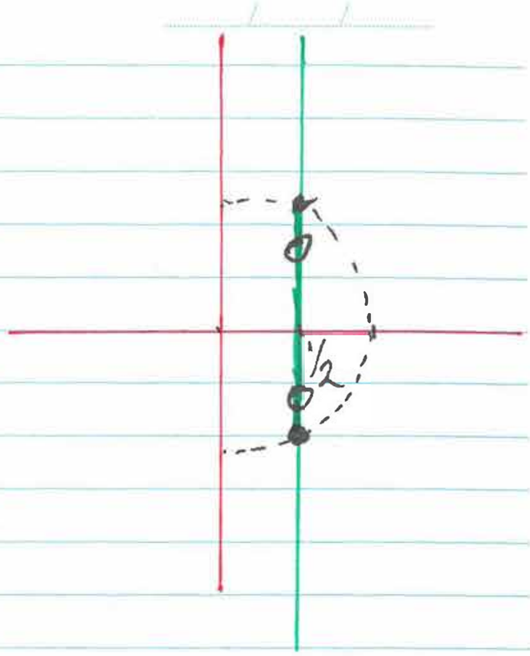
$\times (u^2+v^2)$: $\boxed{A + Bu - Cv + D(u^2+v^2) = 0}$ $\quad r' = \frac{\sqrt{B^2+C^2-4AD}}{2|D|}$

compare with $(**)$!

$(A, D) \neq (0,0)$

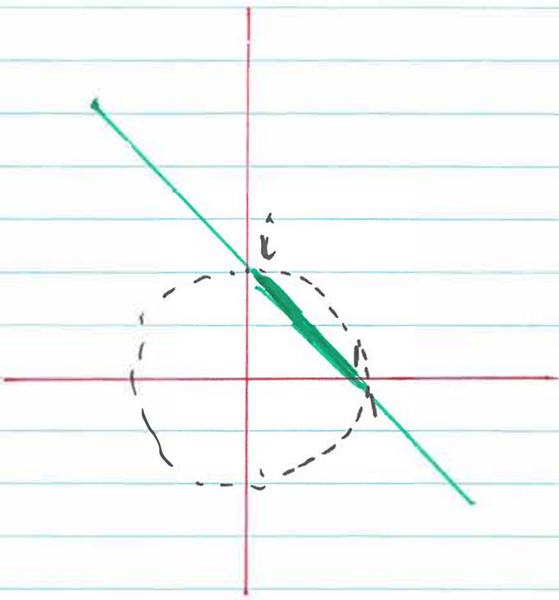
$\Rightarrow \frac{r}{r'} = \frac{|D|}{|A|}$

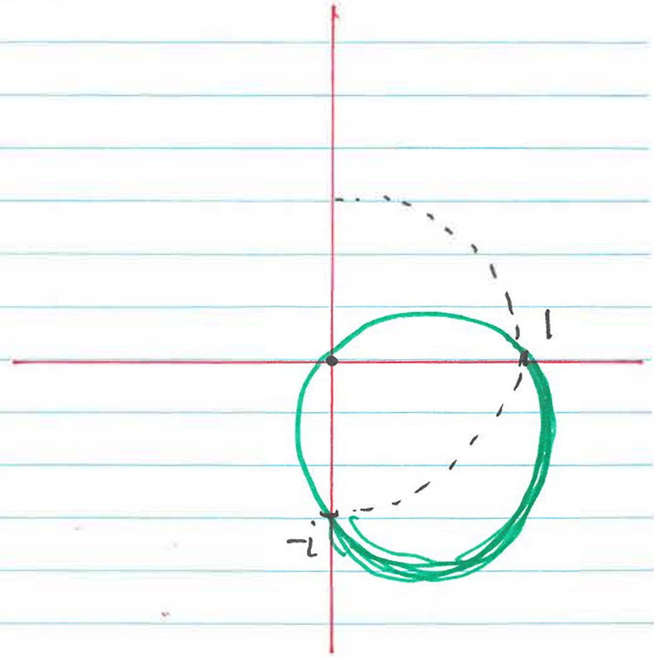




$z = 1 \pm i$

$\mapsto \frac{1 \mp i}{2}$





Möbius transformations / fractional linear transformations

Let $a, b, c, d \in \mathbb{C}$ s.t. $ad - bc \neq 0$.

Then $f(z) = \dfrac{az+b}{cz+d}$ is called a Möbius ~~trafo~~ transformation.

$$f : \mathbb{C} \setminus \left\{ -\frac{d}{c} \right\} \to \mathbb{C}$$

Aside (need this on Friday)

* $f : \Omega \to \mathbb{C}$ is 1-1 or injective if $f(z_1) = f(z_2)$ implies $z_1 = z_2$

* $f : \Omega \to \Omega'$ is onto or surjective if $\forall\, w \in \Omega'$ $\exists\, z \in \Omega$ s.t. $f(z) = w$.

* $f : \Omega \to \Omega'$ is bijective if it is injective & surjective

<u>Homework</u>: What Möbius?