LECTURE 6

For a general Möb transformation

$$T(z) = w = \frac{az+b}{cz+d} \qquad ad-bc \neq 0 \quad (*)$$

(*) can be rewritten in the form

$$Azw + Bz + Cw + D = 0 \quad (**)$$

(with $A = c$, $B = -a$, $C = d$, & $D = -b$), called the implicit form.

Claim: Möbius transforms are 1-1 & onto.

Case I $c = 0$ from Lec. 5,

T is a bijection (1-1 & onto), $\mathbb{C} \to \mathbb{C}$.

Case II $c \neq 0$ argue by the composition from Lec 5, or note that $(*) \Rightarrow z = \dfrac{-dw+b}{cw-a}$

i.e. $T^{-1}(w) = \dfrac{-dw+b}{cw-a}$.

$\Rightarrow$ in case II, T is 1-1 & onto,

$$\mathbb{C} \setminus \{-d/c\} \to \mathbb{C} \setminus \{a/c\}.$$

Q^n: can we extend T to a f^n $\mathbb{C} \to \mathbb{C}$ in case II?

Ans: yes, e.g., take $T(-d/c) = a/c$.

(indeed, taking $T(-d/c) =$ "any fixed value in $\mathbb{C}$" answers the question with "yes") but taking $T(-d/c) = a/c$ makes the extension 1-1 and onto.

Deeply unsatisfying, because this extension is discontinuous at $-d/c$ (intuitively clear, because $|w| \to \infty$ as $z \to -d/c$, but $|T(-d/c)| = |a/c|$, which is finite: will be made rigorous later.

IMPORTANT CONCEPT

Extend $\mathbb{C}$ to the extended complex plane denoted by $\overline{\mathbb{C}}$, by "adding a point at infinity", which we call ∞. So $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

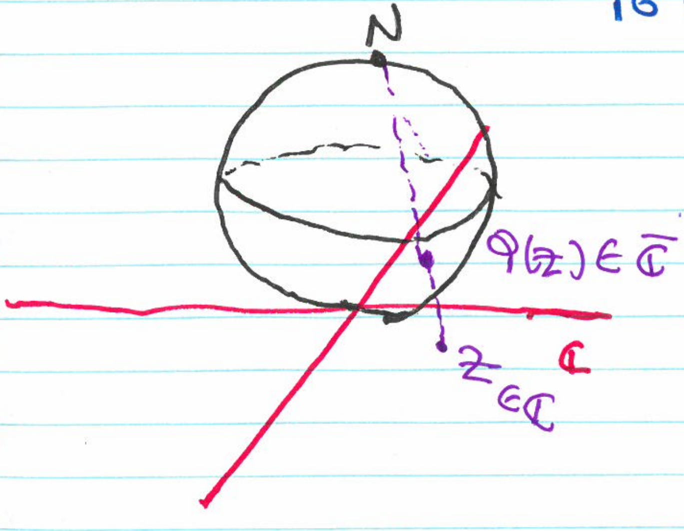
Define $T(-d/c) = \infty$ & $T(\infty) = a/c$.

This extends T to a map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which is 1-1 & onto.

RMK 1 $\overline{\mathbb{C}}$ is a topological space, & the given extension is continuous.

RMK 2 In case I ($c=0$), extend T to a map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by setting $T(\infty) = \infty$.

To visualise & calculate in $\overline{\mathbb{C}}$:



φ maps $\mathbb{C}$ onto the (surface of the) sphere, called the Riemann sphere.

$\varphi(\infty) = N$ extends this to a map (1-1, onto) on $\overline{\mathbb{C}}$.

RMK on Möb transforms:

$$w = \frac{az+b}{cz+d} = \frac{(\lambda a)z+(\lambda b)}{(\lambda c)z+(\lambda d)} \quad \forall \lambda \in \mathbb{C}_*$$

i.e., representation of a Möb transf is only unique up to multiplicative constant $\times$ coefficients.