Math 3401/3901 Lecture 6

Recall that

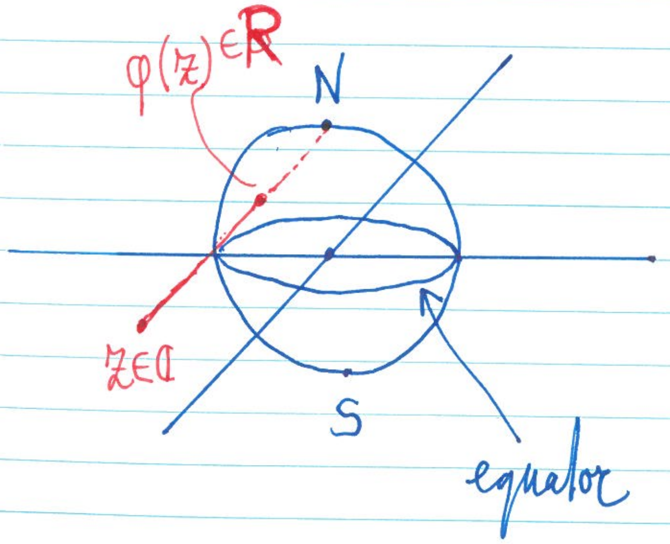
$$w = f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \text{ is called a}$$

Möbius transformation.

In this lecture we will view this fcn as a fcn from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. is called the extended complex plane

To do so, we set $f\left(-\frac{d}{c}\right) = \infty$, $f(\infty) = \frac{a}{c}$, with the understanding that this says that $f(\infty) = \infty$ if $c = 0$.

Topologically & geometrically this corresponds to what is known as the Riemann sphere ($\mathbb{R}$) or complex projective space $\mathbb{C}P^1$



* the unit circle in $\mathbb{C}$ corresponds to the equator on $\mathbb{R}$
* point on the interior of the unit circle corresponds to the southern hemisphere. (~~0 corresponds~~ $\varphi(0) = S$)
* point on the exterior of the unit circle correspond to the punctured (at N) northern hemisphere
* $\varphi(\infty) = N$

Equivalently we represent point in $\mathbb{CP}^1$ as elements of $\mathbb{C}^2 \setminus \{(0,0)\}$ s.t. $\underbrace{(z_1, z_2)}_{\text{"think } \frac{z_1}{z_2}\text{"}} \overset{(\sim)}{=} (\lambda z_1, \lambda z_2)$, $\underset{\in \mathbb{C}}{\lambda} \neq 0$

~~Then~~

Then $z \in \mathbb{C}$ corresponds to $(z, 1)$

& $\infty \in \overline{\mathbb{C}}$ (not in $\mathbb{C}$) $(1, 0)$

Remarks * $w = \dfrac{az+b}{cz+d}$ and $w = \dfrac{(a\lambda)z + (b\lambda)}{(c\lambda)z + (d\lambda)}$

$\lambda \in \mathbb{C} \setminus \{0\}$

represent the same MT.

* If $\underset{(c,d \neq 0)}{ad = bc}$ then $w = \frac{a}{c} = \frac{b}{d}$

is the constant function. This is not a MT: we are neither injective nor surjective.

Proposition Möbius transformations are bijections on $\overline{\mathbb{C}}$

Pf Using projective space, we have

① $z \in \mathbb{C}$, $z \cong (z,1)$

$$\underbrace{\frac{az+b}{cz+d}}_{\in \overline{\mathbb{C}}} \cong (az+b, \underbrace{cz+d}_{\text{could be } 0}) = (w_1, w_2) \neq (a,c)$$

Hence $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = A \begin{pmatrix} z \\ 1 \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(A) \neq 0$: A is invertible

② $\infty \in \overline{\mathbb{C}} \setminus \mathbb{C}$, $\infty \cong (1,0)$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \cong \frac{a}{c}$$

□

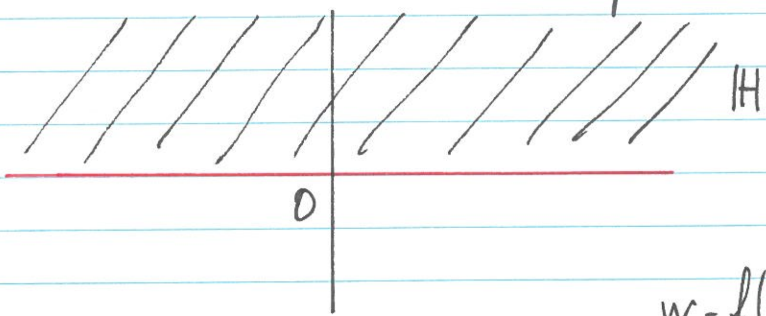
Homework : Show that Circles on R not going through N

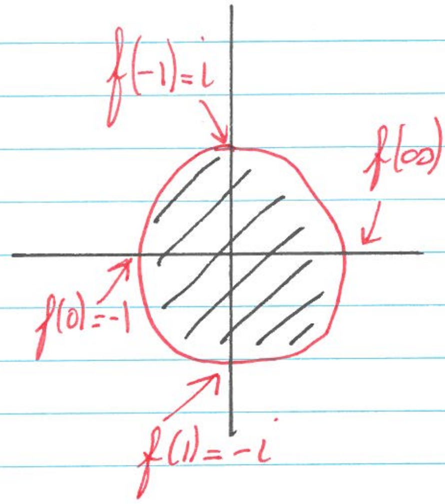
correspond to circles in $\mathbb{C}$

Circles on R throug N correspond to lines in $\mathbb{C}$.

Examples

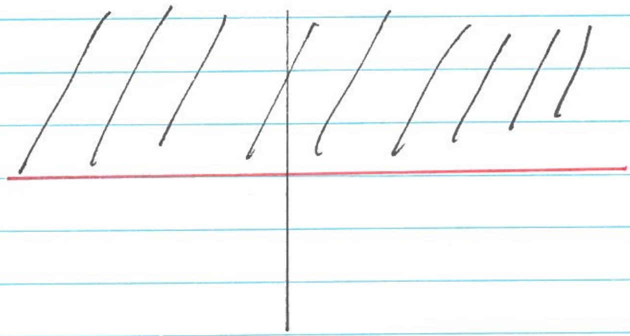
* Find a MT that maps

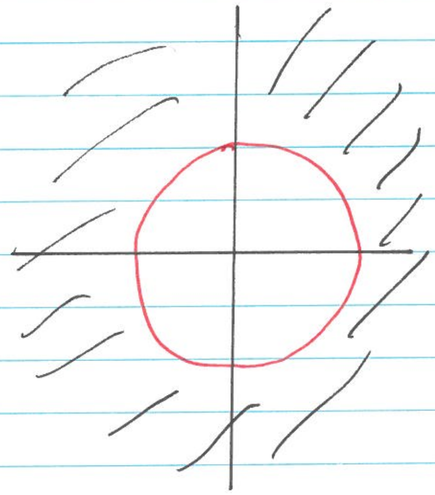




$$w = f(z) = \frac{z-i}{z+i}$$

* Find a MT that maps





$$w = f(z) = \frac{z+i}{z-i}$$

(Recall that $z \mapsto \frac{1}{z}$ sends the interior of the unit circle to the exterior & vice versa).