LECTURE 7

Exercise: prove $|z| = |\bar{z}|$ for $z \in \mathbb{C}$.

① Let $z = x + iy$.

$$LHS = \sqrt{x^2+y^2} = \sqrt{x^2+(-y)^2} = |x - iy| = RHS.$$

OR

②
$$|z|^2 = z\bar{z}$$
$$= \bar{\bar{z}}\ \bar{z}$$
$$= |\bar{z}|^2.$$

take $\sqrt{\ }$ $\Rightarrow$ result.

Final Remarks on Möbius transforms.

RMK ① Given 3 distinct points in $\overline{\mathbb{C}}$, z_1, z_2 & z_3 & 3 distinct points in $\mathbb{C}$ w_1, w_2 & w_3

$\exists!$ Möb. transf. T s.t.

there exists a unique

$$T(z_j) = w_j, \quad j = 1, 2, 3.$$

$T = w(z)$ is given by:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad (**)$$

Note ①: in practice, may be easier to solve directly for a, b, c, d rather than using $(**)$

Note ②: how does this work with ∞?

$T(\infty) = a/c$ & $T(-d/c) = \infty$;

$T(\infty) = \infty \Rightarrow c = 0.$

rigorously later:

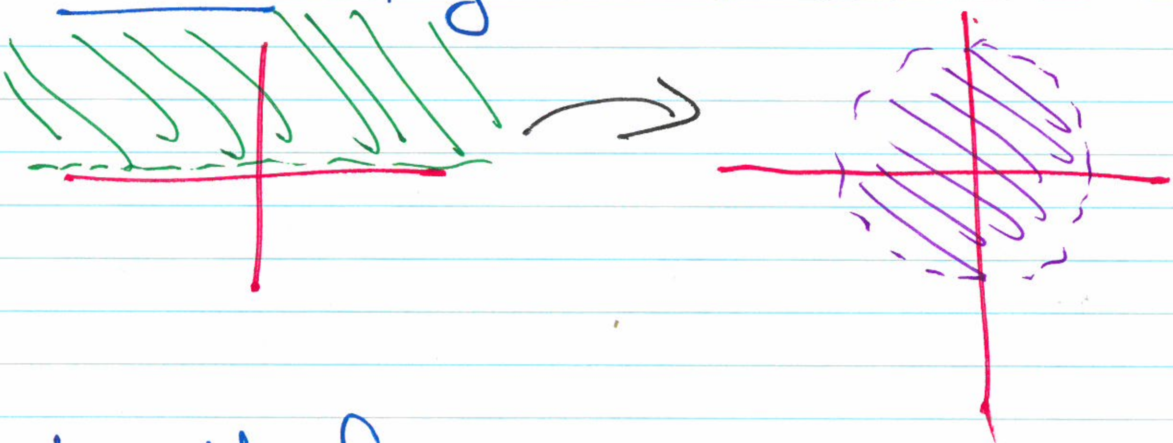
$$T(\infty) = w \iff \lim_{|z|\to\infty} T(z) = w \,;$$

$$T(\xi) = \infty \iff \lim_{z\to\xi} \frac{1}{T(z)} = 0 .$$

RMK ② $$w = \frac{az+b}{cz+d} = \frac{(\lambda a)z + (\lambda b)}{(\lambda c)z + (\lambda d)} \quad \forall \lambda \in \mathbb{C}_*$$

i.e., representation of a Möb transf. is only unique up to multiplicative constant × coefficients.

RMK ③ Any Möb transf.



has the form

$$w = e^{i\alpha}\frac{z-z_0}{z-\bar{z}_0} \quad \text{for some}$$

$\alpha \in \mathbb{R}$ & $z_0 \in \mathbb{C}$, with $\mathrm{Im}(z_0) > 0$, & any Möb transf. of this form maps the UHP (upper half plane) onto the inside of the unit circle.

§103 (8 Ed §104) Exponential Map.

$z \mapsto e^z = \exp z = w$

$\mathrm{dom}(w) = \mathbb{C}$.

$z = x + iy \quad x, y \in \mathbb{R} \quad w = e^z = e^{x+iy} = e^x e^{iy}$

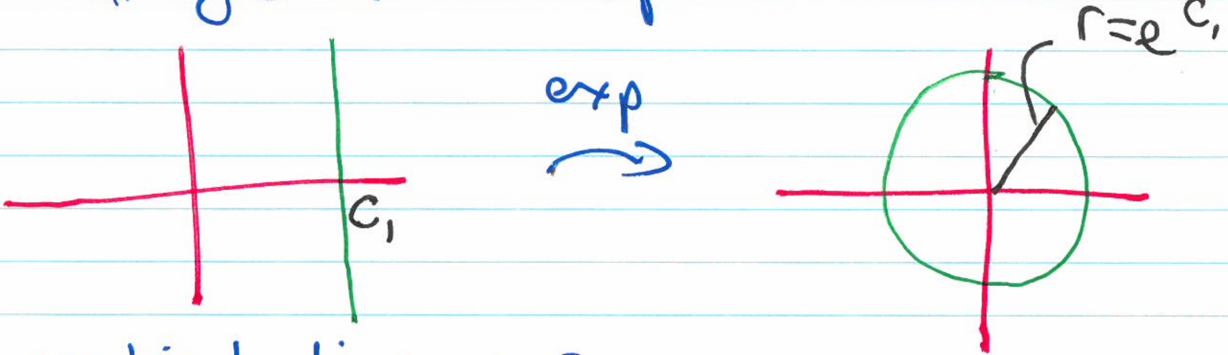
$= e^x(\cos y + i \sin y) = u + iv$, where

$u = e^x \cos y$ & $v = e^x \sin y$.

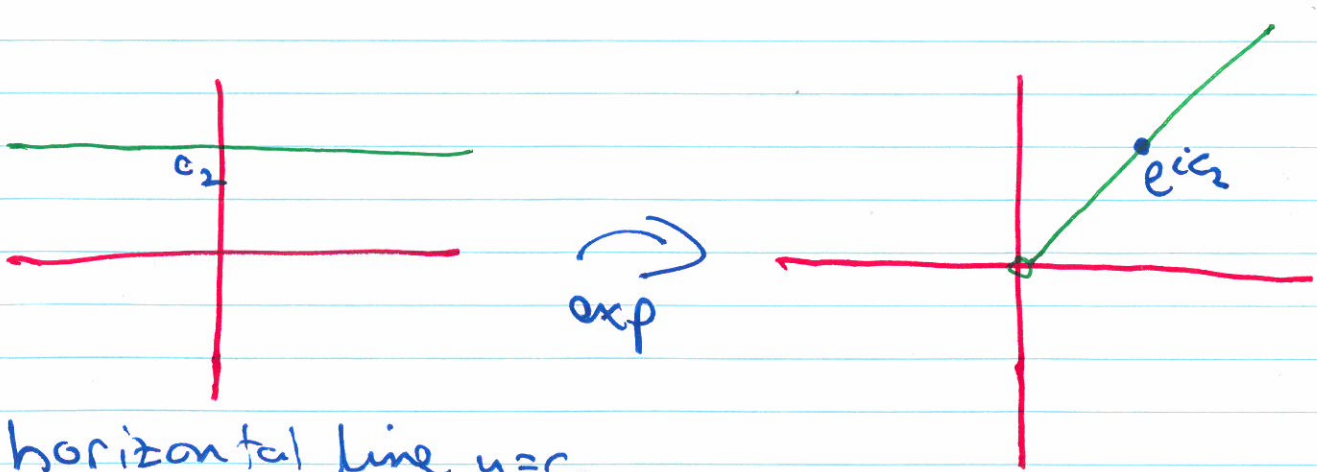
Write $w = \rho e^{i\phi}$, where

$$\begin{cases} \rho = e^x \\ \phi = y + 2k\pi \quad k \in \mathbb{Z}. \end{cases}$$

Images under exp:



vertical line $x = c_1$



horizontal line $y = c_2$

How about :