LECTURE 7.

Final remarks on Möb. transf.

RMK 1 Given 3 distinct points in $\bar{\mathbb{C}}$, z_1, z_2 & z_3, & 3 distinct points in $\bar{\mathbb{C}}$, w_1, w_2 & w_3, then

$\exists!$ Möb. transf T s.t.

there exists a unique

$$T(z_j) = w_j \qquad j = 1,2,3.$$

$T = w(z)$ is given by:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad (**)$$

Note ①: in practice, often easier to solve directly for a,b,c,d rather than using (**)

Note ②: how does this work with ∞?

$$T(\infty) = a/c \quad ; \quad T(-d/c) = \infty \; ;$$

$$T(\infty) = \infty \iff c = 0.$$

rigorously later:

$$T(\infty) = w \iff \lim_{|z| \to \infty} T(z) = w;$$

$$T(\zeta) = \infty \iff \lim_{z \to \zeta} 1/T(z) = 0.$$

RMK (2)

Any Möb transf.

has the form

$$w = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}$$

for some $\alpha \in \mathbb{R}$ & $z_0 \in \mathbb{C}$ with $\mathrm{Im}(z_0) > 0$, & any Möb transf of this form maps the UHP (upper half plane) onto the inside of the unit circle.

§103 (8 Ed §104) Exponential Map.

$z \mapsto e^z = \exp z = w$

$\mathrm{dom}(w) = \mathbb{C}$

$z = x+iy \quad x,y\in\mathbb{R} \quad w = e^z = e^{x+iy} = e^x e^{iy}$

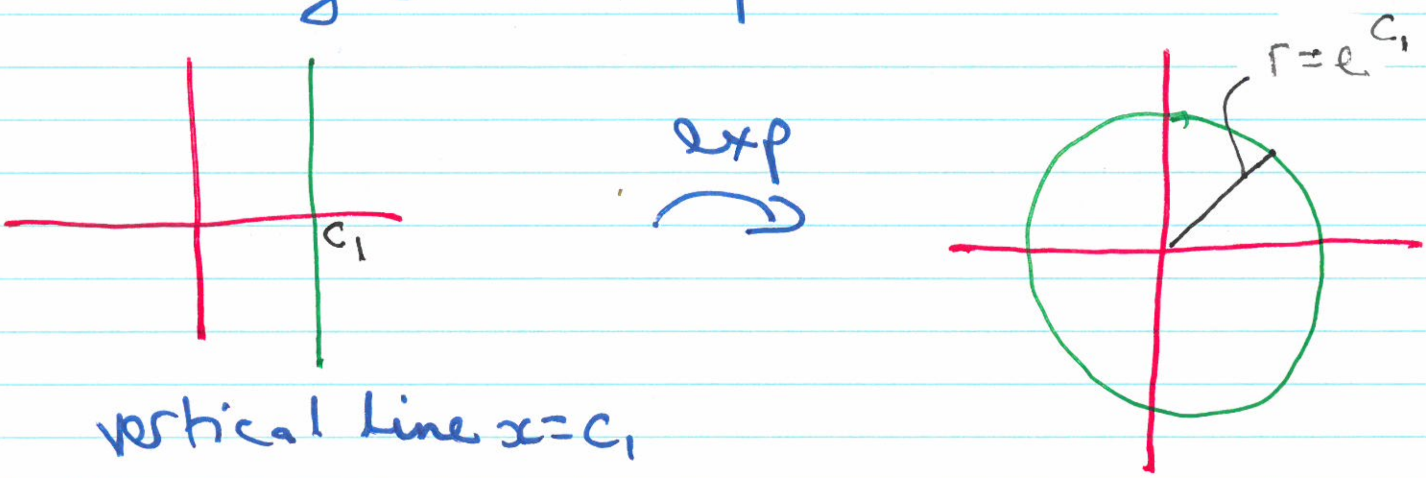
$= e^x(\cos y + i \sin y) = u + iv$ where

$u = e^x \cos y$ & $v = e^x \sin y$.

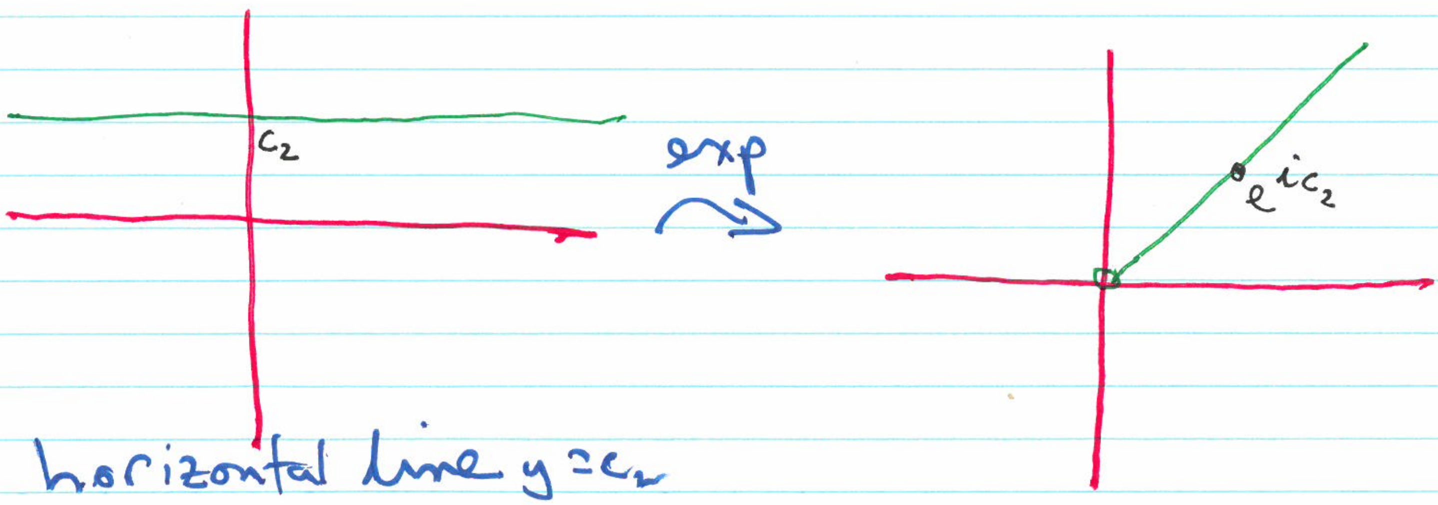
Write $w = \rho e^{i\phi}$, where

$$\begin{cases} \rho = e^x \\ \phi = y + 2k\pi & k \in \mathbb{Z} \end{cases}.$$

Images under exp :

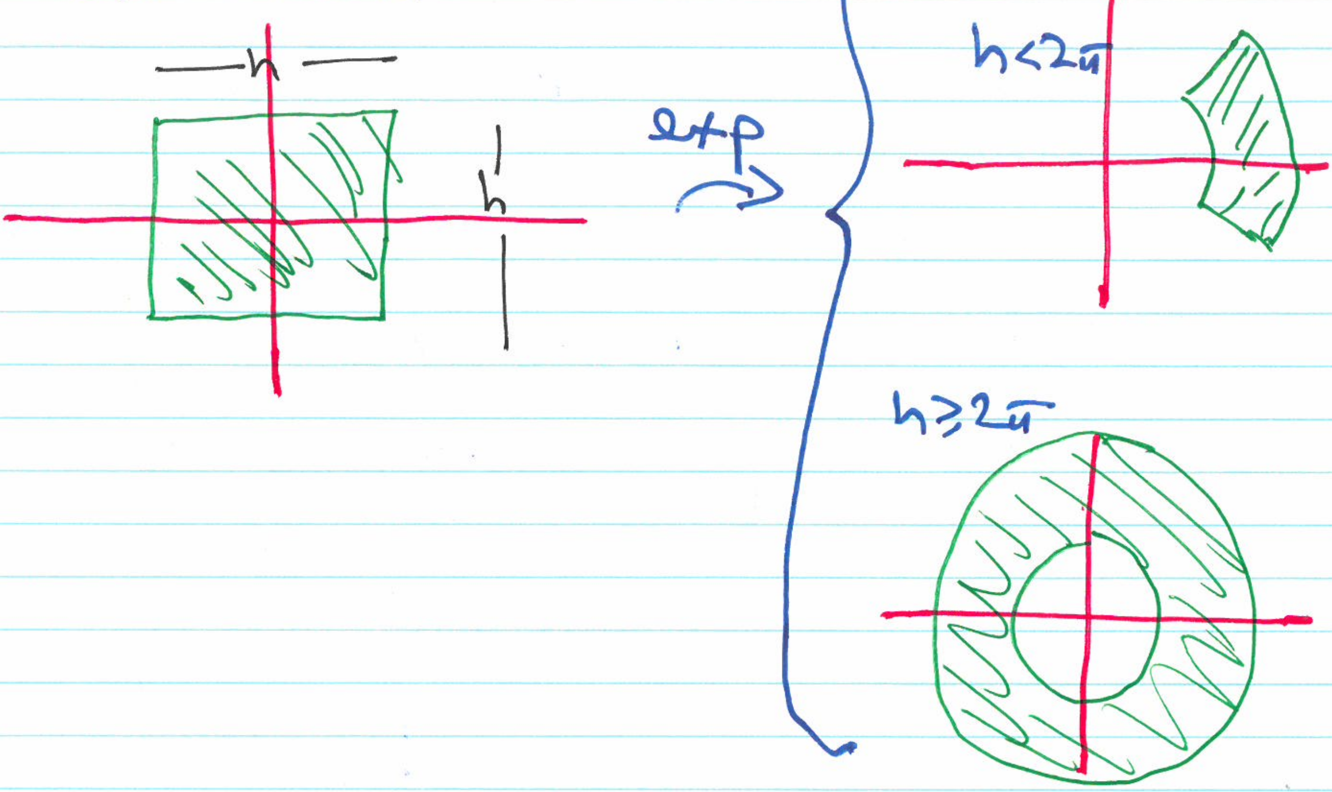


vertical line $x = c_1$



horizontal line $y = c_2$

How about:



§30 (8 Ed §29)

Many "laws" for $\mathbb{R}$-exp extend to $\mathbb{C}$-exp using $e^z = e^{x+iy} = e^x e^{iy}$ for $z = x+iy$.

① $e^0 = 1$;

② $e^{-z} = 1/e^z$;

③ $e^{z_1+z_2} = e^{z_1} e^{z_2}$;

④ $e^{z_1-z_2} = e^{z_1}/e^{z_2}$;

⑤ $(e^{z_1})^{z_2} = e^{z_1 z_2}$.

Some things don't extend

⑥ $e^x > 0 \ \forall x \in \mathbb{R}$, but, e.g., $e^{i\pi} = -1$, $e^{i\pi/4} \in \mathbb{C} \setminus \mathbb{R}$.

⑦ $x \mapsto e^x$ is monotone increasing in $\mathbb{R}$; but $z \mapsto e^z$ is periodic in $\mathbb{C}$, with period $2\pi i$

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z \overbrace{(\cos 2\pi + i \sin 2\pi)}^{=1} = e^z.$$

Note as in $\mathbb{R}$, $e^z = 0$ has no solⁿ in $\mathbb{C}$.

If $\exists z = x+iy \in \mathbb{C}$ s.t. $e^z = 0$ then

$$e^x \underbrace{e^{iy}}_{\mathbb{C}\text{ number of modulus } 1} = 0 \Rightarrow e^x = 0 \quad C!$$

Contradiction.