LECTURE 7.

Semi-final remarks on Möbius transformations.

RMK ① Given 3 distinct points in $\overline{\mathbb{C}}$, z_1, z_2 & z_3, & 3 distinct points in $\overline{\mathbb{C}}$, w_1, w_2, w_3

$\exists!$ Möb transf. T, s.t.,

there exists a unique

$$T(z_j) = w_j \qquad j = 1, 2, 3.$$

$T = w(z)$ is given by:

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \qquad **$$

Note ① : in practice, may be (usually is) easier to solve directly for a, b, c, d rather than just using **.

Note ② : how does this work with ∞?

$$T(\infty) = a/c, \quad T(-d/c) = \infty.$$

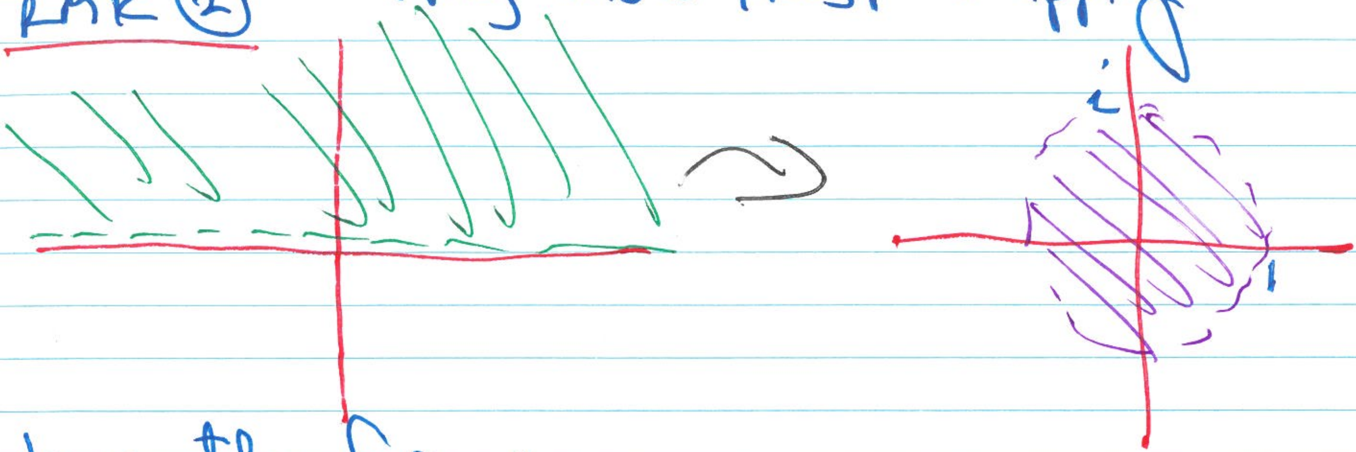
$$T(\infty) = \infty \Rightarrow c = 0.$$

rigorously later:

$$T(\infty) = w \Leftrightarrow \lim_{z \to \infty} T(z) = w;$$

$$T(\xi) = \infty \Leftrightarrow \lim_{z \to \xi} 1/T(z) = 0.$$

RMK (2) Any möb transf mapping



has the form:

$$w = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}$$ for some $z_0 \in \mathbb{C}$ with

$\mathrm{Im}(z_0) > 0$ & $\alpha \in \mathbb{R}$, & any Möb transf. of this form maps the UHP (upper half plane) onto the inside of the unit circle.

§103 (8Ed §104) Exponential Map.

$z \mapsto e^z = \exp z = w.$

$\mathrm{dom}(w) = \mathbb{C}.$

$z = x+iy \qquad x, y \in \mathbb{R}: \; w = e^z = e^{x+iy} = e^x e^{iy}$

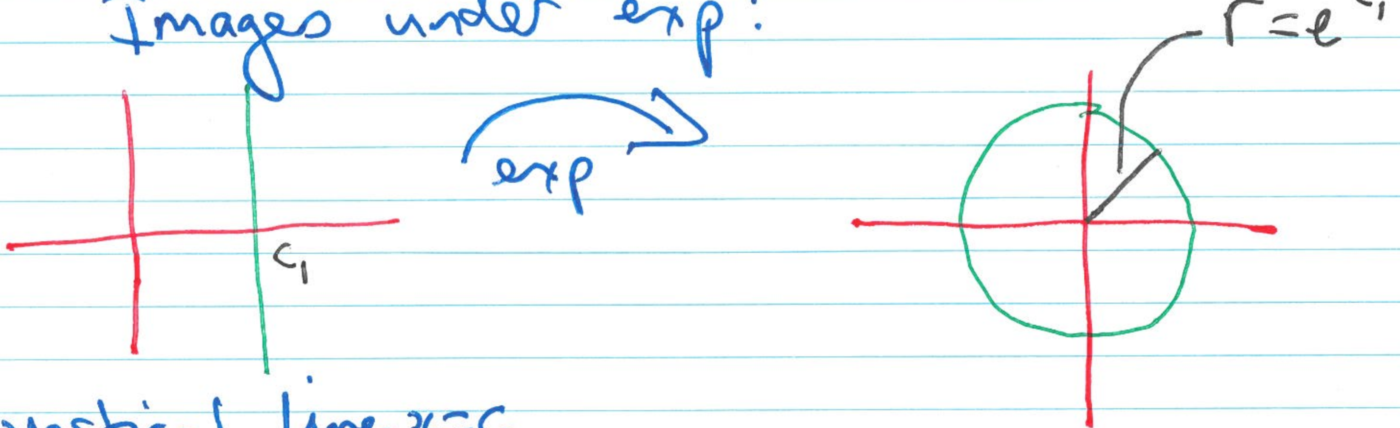
$= e^x(\cos y + i \sin y) = u + iv$, where

$u = e^x \cos y, \; v = e^x \sin y.$

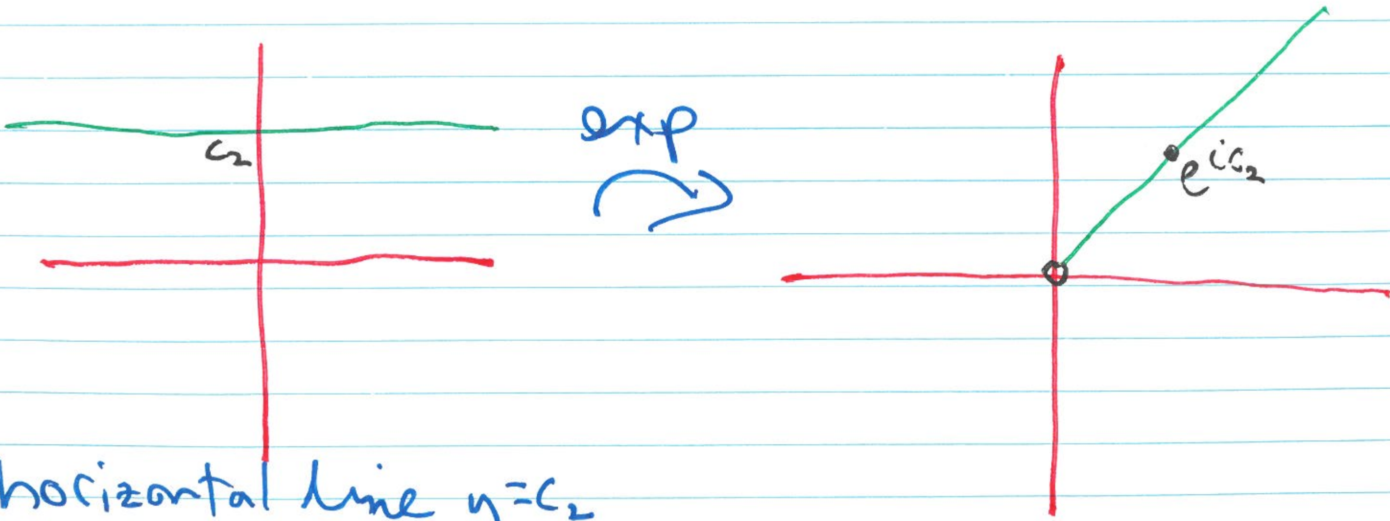
Write $w = \rho e^{i\phi}$, where

$$\begin{cases} \rho = e^x \\ \phi = y + 2k\pi & k \in \mathbb{Z}. \end{cases}$$

Images under exp:

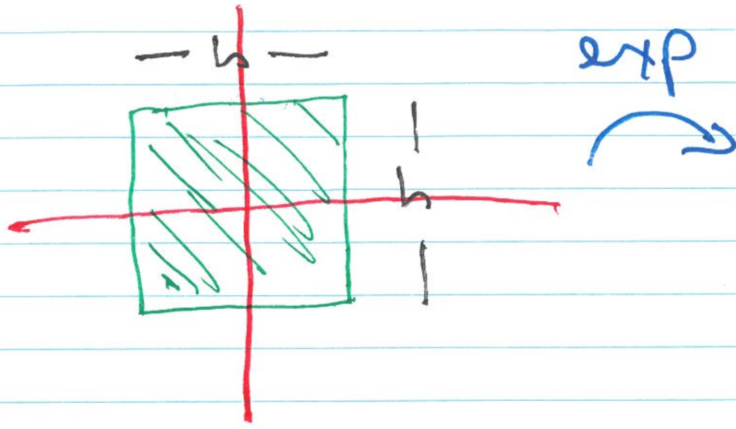


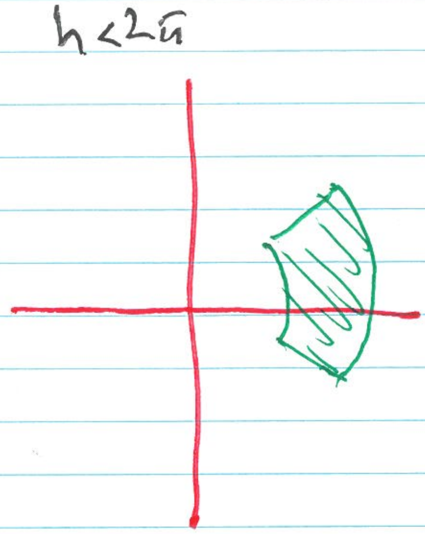
vertical line $x = c_1$

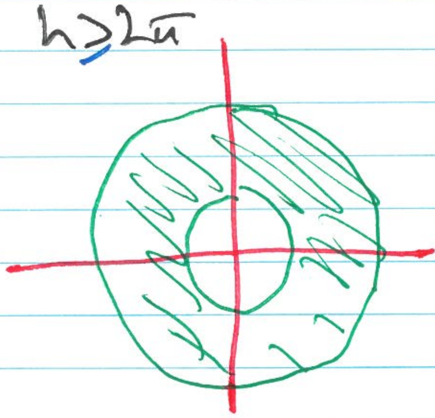


horizontal line $y = c_2$

How about:







§30 (& end §29)

Many "laws" for $\mathbb{R}$-exp extend to $\mathbb{C}$-exp, using $e^z = e^{x+iy} = e^x e^{iy}$ for $z = x+iy$.

(1) $e^0 = 1$;

(2) $e^{-z} = 1/e^z$;

(3) $e^{z_1+z_2} = e^{z_1} e^{z_2}$;

(4) $e^{z_1-z_2} = e^{z_1}/e^{z_2}$;

(5) $(e^{z_1})^{z_1} = e^{z_1 z_2}$

Some things don't extend:

(6) $e^x > 0\ \forall x \in \mathbb{R}$, but, e.g., $e^{i\pi} = -1$, & $e^{i\pi/4} \in \mathbb{C}\setminus\mathbb{R}$;

(7) $x \mapsto e^x$ is monotone increasing on $\mathbb{R}$, but $z \mapsto e^z$ is periodic with period $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i\sin 2\pi) = e^z$$

Note as in $\mathbb{R}$, $e^z = 0$ has no sol^n in $\mathbb{C}$.

If $\exists z = x+iy$ s.t. $e^z = 0$, then

$$e^x \underbrace{e^{iy}}_{\mathbb{C} \text{ number of modulus } 1} = 0 \Rightarrow e^x = 0 \quad C!$$

contradiction.