LECTURE 8

§30 (8Ed §29)

Many "laws" for $\mathbb{R}$-exp extend to $\mathbb{C}$-exp, using $e^z = e^{x+iy} = e^x e^{iy}$ for $z = x+iy$.

① $e^0 = 1$;

② $e^{-z} = 1/e^z$;

③ $e^{z_1+z_2} = e^{z_1} e^{z_2}$;

④ $e^{z_1-z_2} = e^{z_1}/e^{z_2}$;

⑤ $(e^{z_1})^{z_2} = e^{z_1 z_2}$.

Some things don't extend

⑥ $e^x > 0 \ \forall x \in \mathbb{R}$, but, e.g., $e^{i\pi} = -1$, & $e^{i\pi/4} \in \mathbb{C} \setminus \mathbb{R}$.

⑦ $x \mapsto e^x$ is monotone increasing in $\mathbb{R}$; but $z \mapsto e^z$ is periodic in $\mathbb{C}$, with period $2\pi i$: $e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z$.

Note: as in $\mathbb{R}$, $e^z = 0$ has no sol^{n} in $\mathbb{C}$.

If $\exists z = x+iy$ s.t. $e^z = 0$, then

$$e^x \underbrace{e^{iy}}_{\mathbb{C}\text{ number of modulus } 1.} = 0 \Rightarrow e^x = 0 \quad \text{C!}$$

contradiction.

Inverses §31–33 (8Ed §30–32).

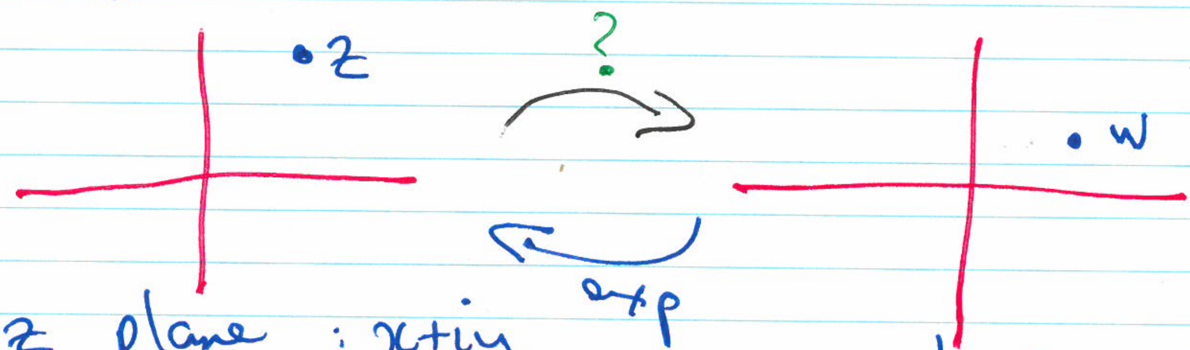
$f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$

Then $g : \text{Range}(f) \to \Omega$ is an <u>inverse</u> of f say that $g \circ f : \Omega \to \Omega$ is the identity, i.e., $(g \circ f)(z) = z \quad \forall z \in \Omega$.

E.g., $z \mapsto z+1$, $z \mapsto z-1$ are inverses of each other, $\mathbb{C} \to \mathbb{C}$;
$z \mapsto \frac{1}{z}$ is its own inverse, $\mathbb{C}_* \to \mathbb{C}_*$.

Inverse of exp?

Note: no universal convention for log vs $\log_e$ vs ln.



z plane : $x+iy$

w plane : $u+iv$

Too much to hope for "$? = \log = \log_e$",
because exp is periodic in $\mathbb{C}$.

Note $e^w = z$

write $z = re^{i\Theta}$, $r > 0$

$\Theta = \operatorname{Arg} z \quad (\Theta \in (-\pi, \pi])$

Write $w = u + iv$

$$\Rightarrow \quad \begin{array}{l} e^w = e^{u+iv} = e^u e^{iv} \\ \parallel \\ z = re^{i\Theta} \end{array}$$

$$\Rightarrow \begin{cases} e^u = r \\ v = \Theta + 2k\pi \quad k \in \mathbb{Z}. \end{cases}$$

So $\boxed{u = \ln r}$ meaning (in BC, & in this course) logarithm to the base e of the +ve real number r.

So $w = u + iv$

$= \ln r + i(\Theta + 2k\pi) \quad k \in \mathbb{Z}$

$= \ln |z| + i \arg z.$

This defines the mult-valued f^n log on $\mathbb{C}_*$.

Note ① $\exp(\log z) = z$ &

② $\log(\exp z) = z + 2k\pi i$, $k \in \mathbb{Z}$ ✱

① $\exp(\log z)$ $\quad z \neq 0$

$= \exp(\ln|z| + i \arg z)$

$= \exp(\ln|z| + i(\theta + 2k\pi))$ $\forall k \in \mathbb{Z}$ for some fixed argument

$= \exp(\ln|z| + i\theta + 2k\pi i)$. θ of z (e.g. $\operatorname{Arg} z$).

$= \exp(\ln|z| + i\theta) \cdot \exp 2k\pi i$

$= \exp(\ln|z| + i\theta) \cdot 1$

$= \exp(\ln|z| + i\theta)$

$= |z| e^{i\theta} = z$.

Check $\log(z\zeta) = \log z + \log \zeta$;

$\log(z/\zeta) = \log z - \log \zeta$; etc.

As with Arg/arg, define

$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ on $\mathbb{C}_*$

Log is single valued, but discts on $-$ve Re axis $\cup \{0\}$, since Arg is (indeed, Arg/Log aren't even defined at 0).

Log is called the principal logarithm.

As with Arg, there may hold:

$\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$ etc.