LECTURE 8

Inverses §31-33 (8Ed. §30-32).

$f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$: Then $g : \text{Range}(f) \to \Omega$ is an inverse of f says that $g \circ f : \Omega \to \Omega$ is the identity, i.e.,

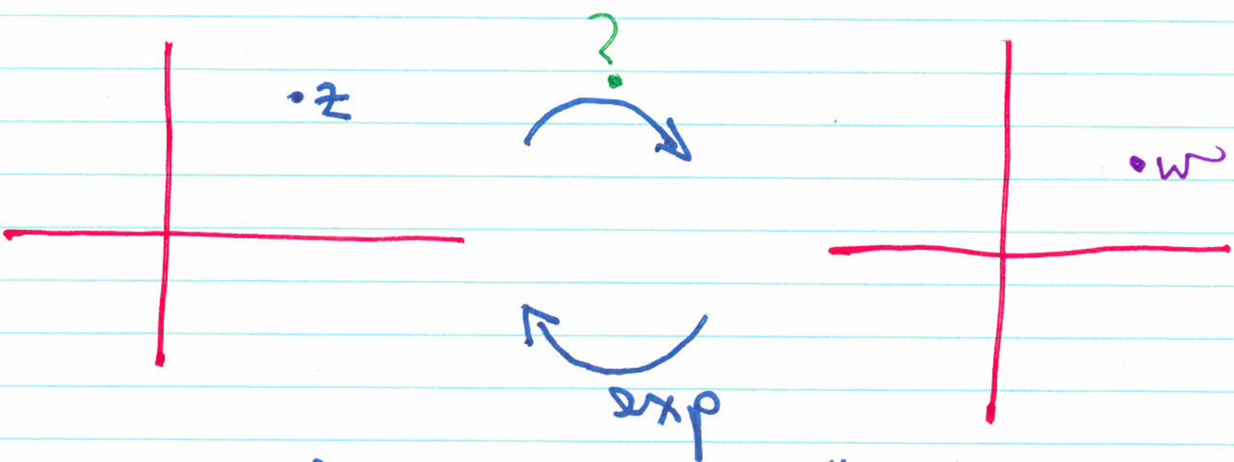
$$(g\circ f)(z) = z \qquad \forall z \in \Omega$$

E.g. * $z \mapsto z+1$, $z \mapsto z-1$ are inverses of each other, $\mathbb{C} \to \mathbb{C}$.

* $z \mapsto 1/z$ is its own inverse, $\mathbb{C}_* \to \mathbb{C}_*$

> Can we find an inverse for exp / can we find a f^n whose inverse is exp?

Note: no universal convention for log vs $\log_e$ vs ln. Different books can & will have different conventions.



Too much to hope for "? = log = $\log_e$" because exp is periodic in $\mathbb{C}$.

Note $e^w = z$

write $z = re^{i\Theta}$ $r > 0$

$\Theta = \operatorname{Arg} z$ (so $\Theta \in (-\pi, \pi]$)

Write $w = u + iv$

$\Rightarrow e^w = e^{u+iv} = e^u e^{iv}$

$\parallel$

$z = re^{i\Theta}$

$\Rightarrow e^u = r$

$v = \Theta + 2k\pi$ $k \in \mathbb{Z}$.

So $\boxed{u = \ln r}$ meaning (in BC, & hence in MATH3401/3901) logarithm to the base e of the +ve real number r.

So $w = u + iv$

$= \ln r + i(\Theta + 2k\pi)$ $k \in \mathbb{Z}$

$= \ln |z| + i \arg z$.

This defines the multi-valued function log on $\mathbb{C}_*$.

Note ① $\exp(\operatorname{Log} z) = z$ ∀ $z \neq 0$

* ② $\operatorname{Log}(\exp z) = z + 2k\pi i$ $k \in \mathbb{Z}$

Re ① $\exp(\operatorname{Log} z)$ $z \neq 0$

$= \exp(\ln|z| + i \arg z)$

$= \exp(\ln|z| + i(\theta + 2k\pi))$ $\forall k \in \mathbb{Z}$ for some fixed argument θ of z, e.g. $\operatorname{Arg} z$.

$= \exp(\ln|z| + i\theta + i2k\pi)$

$= \exp(\ln|z| + i\theta) \cdot \exp(i2k\pi)$

$= \exp(\ln|z| + i\theta) \cdot 1$

$= \exp(\ln|z| + i\theta)$

$= |z|e^{i\theta} = z.$

cf. inverse trig f^{n}s:

$\sin^{-1}(\sin \pi/4) = \pi/4$ but $\sin^{-1}(\sin 9\pi/4) = \pi/4$.

Check: $\log(z\zeta) = \log z + \log \zeta$

$\log(z/\zeta) = \log z - \log \zeta$ etc.

As with Arg/arg, define

$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ on $\mathbb{C}_*$

Log is single valued, but is discts. on -ve Re axis $\cup \{0\}$, since Arg is (indeed, Arg/Log aren't even defined at 0.

Log is called the principal logarithm.

As with Arg, there may hold:

$$\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 .$$

Complex exponentials:

Analogously to the situation in $\mathbb{R}$, set $z^c = \exp(c \,\mathrm{Log}\, z)$, $z \neq 0$.

Note: corresponds to the usual index laws for $c = n \in \mathbb{Z}$, & $c = 1/n$ $n \in \mathbb{Z}$ we recover what we did in Lec 3 for nth roots.

RMK: BC defines $z^{1/n}$ as a multi-valued f^n, & in particular, defines

$$PV(z^{1/n}) = |z|^{1/n} \exp\left(\frac{i \,\mathrm{Arg}\, z}{n}\right).$$

principal value

Same procedure works for $z \mapsto z^c$:

$$\begin{aligned} PV(z^c) &= \exp(c \operatorname{Log} z) \\ &= \exp(c(\ln|z| + i \operatorname{Arg} z)). \end{aligned}$$

E.g. $PV\left[\underbrace{(1-i)}_{=z}{}^{\overbrace{4i}^{=c.}}\right]$

$$\begin{aligned} &= \exp[4i \operatorname{Log}(1-i)] \\ &= \exp[4i(\ln|1-i| + i \operatorname{Arg}(1-i))] \\ &= \exp[4i(\ln\sqrt{2} - i\pi/4)] \\ &= \exp(4i\ln\sqrt{2} + \pi) \\ &= \exp(\pi + (2\ln 2)i) \\ &= e^{\pi}(\cos(2\ln 2) + i\sin(2\ln 2)). \end{aligned}$$