LECTURE 8

P.S. regarding the "3 point" formula for a Möb. transform (i.e., $T(z_j) = w_j$ for given distinct z_j, w_j, $j = 1,2,3$).

3 eq^n's in 4 unknowns, but

since $\dfrac{az+b}{cz+d}$ & $\dfrac{(\lambda a)z + (\lambda b)}{(\lambda c)z + (\lambda d)}$

are the same Möb transfs for $\lambda \neq 0$,

you can fix any one of the constants, keeping in mind the constraint $ad - bc \neq 0$.

Inverses: §31–33 (8 Ed §30–32)

$f : \Omega \to \mathbb{C}$, $\Omega \to \mathbb{C}$. Then

$g : \text{Range}(f) \to \Omega$ is an inverse of f

says that $(g \circ f) : \Omega \to \Omega$ is the identity,

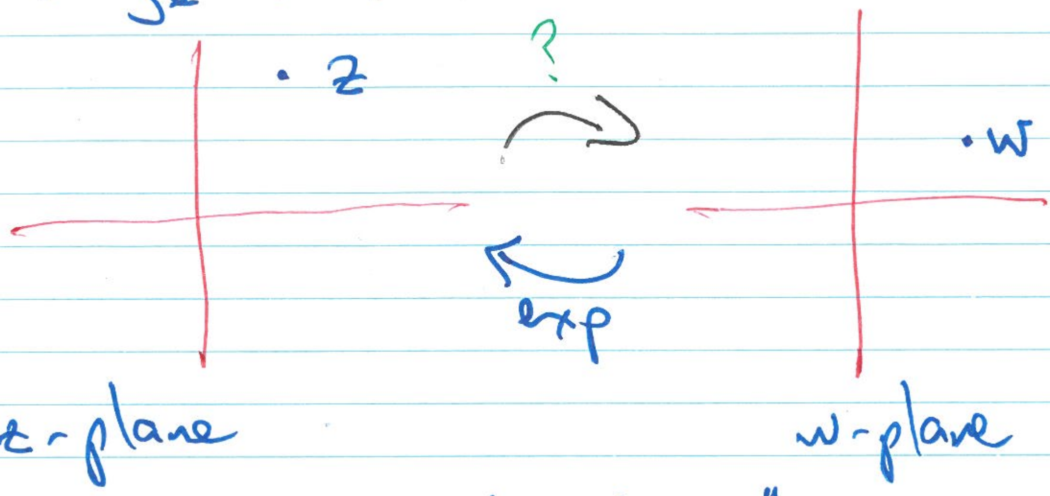
i.e., $(g \circ f)(z) = z$, $\forall z \in \Omega$.

E.g. $z \mapsto z+1$, $z \mapsto z-1$ are inverses of each other, $\mathbb{C} \to \mathbb{C}$;

$z \mapsto 1/z$ is its own inverse, $\mathbb{C}_* \to \mathbb{C}_*$.

Inverse of exp?

Note: no universal convention for log vs $\log_e$ vs ln.



Too much to hope for "$? = \log = \log_e$", because exp is periodic in $\mathbb{C}$.

Note $e^w = z$.

Write $z = re^{i\Theta}$, $r > 0$

$\Theta = \operatorname{Arg} z$ $\quad (\Theta \in (-\pi, \pi])$

Write: $w = u + iv$

$$\Rightarrow \quad e^w = e^{u+iv} = e^u \cdot e^{iv}$$
$$\| $$
$$z = re^{i\Theta}$$

$$\Rightarrow \begin{cases} e^u = r \\ v = \Theta + 2k\pi \quad k \in \mathbb{Z} \end{cases}$$

So, $\boxed{u = \ln r}$, meaning (in BC, & in this course), logarithm to the base e of the +ve real number r.

So $w = u + iv$
$$= \ln r + i(\theta + 2k\pi) \quad k \in \mathbb{Z}$$
$$= \ln|z| + i \arg z .$$
This defines the multi-valued f^n: $\log$ on $\mathbb{C}_*$

Note: ① $\exp(\log z) = z$ &
② $\log(\exp z) = z + 2k\pi i, \ k \in \mathbb{Z}$. ✗

① : $\exp(\log z) \qquad z \neq 0$
$$= \exp(\ln|z| + i \arg z)$$
$$= \exp(\ln|z| + i(\theta + 2k\pi)) \quad \forall k \in \mathbb{Z},$$ for some fixed argument of z, e.g., $\operatorname{Arg} z$.
$$= \exp(\ln|z| + i\theta + 2k\pi i)$$
$$= \exp(\ln|z| + i\theta) \cdot \exp(2k\pi i)$$
$$= \exp(\ln|z| + i\theta) \cdot 1$$
$$= \exp(\ln|z| + i\theta)$$
$$= |z| e^{i\theta} = z .$$

Check: $\log(z\xi) = \log z + \log \xi$;
$\log(z/\xi) = \log z - \log \xi, \ \xi \neq 0$.

As with Arg/arg, define
$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z \quad \text{on } \mathbb{C}_*$$
Log is single-valued, but discts on -ve Real axis $\cup \{0\}$, since Arg is (& Arg/Log aren't even defined at 0).

Complex exponentials:

Analogous to the situation in $\mathbb{R}$, set:

$$z^c = \exp(c \operatorname{Log} z) \quad * \qquad z \neq 0.$$

Note: generalises usual index laws for $c = n \in \mathbb{Z}$, & for $c = 1/n$ ($n \in \mathbb{Z}$) we recover what we did in Lec. 3 for nth roots.

RMK: B-C defines $z^{1/n}$ as a multi-valued f^n, & in particular, defines

$$PV(z^{1/n}) = |z|^{1/n} \exp\left(\frac{i \operatorname{Arg} z}{n}\right).$$

principal value.

Same procedure works for $z \mapsto z^c$:

$$PV(z^c) = \exp(c \operatorname{Log} z)$$

$$= \exp(c(\ln|z| + i \operatorname{Arg} z))$$

E.g. $PV[(1-i)^{4i}]$ $\quad$ $4i \to c$, $\quad (1-i) \to z$

$$= \exp[4i \operatorname{Log} z]$$

$$= \exp[4i(\ln|1-i| + i \operatorname{Arg}(1-i))]$$

$$= \exp[4i(\ln\sqrt{2} - i\pi/4)]$$

$$= \exp[4i \ln\sqrt{2} + \pi]$$

$$= \exp[(2\ln 2) i + \pi]$$

$$= e^{\pi}(\cos(2\ln 2) + i\sin(2\ln 2)).$$