LECTURE 9

Complex exponentials:

Analogously to the situation in $\mathbb{R}$, set:

$$z^c = \exp(c \log z) \qquad * \; z \neq 0.$$

Note: corresponds to the usual index laws for $c = n \in \mathbb{Z}$, & for $c = 1/n$ ($n \in \mathbb{Z}$) we recover what we did in Lec. 3 for nth roots.

RMK B–C defines $z^{1/n}$ as a multi-valued fn, & in particular, defines

$$PV(z^{1/n}) = |z|^{1/n} \exp\left(\frac{i \operatorname{Arg} z}{n}\right).$$

principal value

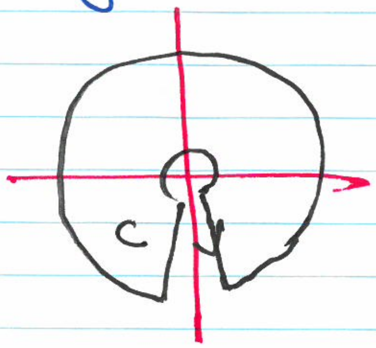
Same procedure works for $z \mapsto z^c$:

$$PV(z^c) = \exp(c \operatorname{Log} z)$$
$$= \exp(c(\ln|z| + i \operatorname{Arg} z))$$

E.g. $PV[(1-i)^{4i}]$ c z

$$= \exp[4i \operatorname{Log}(1-i)]$$
$$= \exp[4i[\ln|1-i| + i \operatorname{Arg}(1-i)]]$$
$$= \exp[4i(\ln\sqrt{2} - i\pi/4)]$$
$$= \exp[4i \ln\sqrt{2} + \pi]$$
$$= \exp[(2\ln 2)i + \pi]$$
$$= e^{\pi}(\cos 2\ln 2 + i \sin 2\ln 2)$$

RMK: sometimes need to use a different single-valued log & arg, e.g. integrating around a contour such as C, called a key-hole contour.



In this case, we would choose/define $\operatorname{Arg} z$ s.t. $-\pi/2 < \operatorname{Arg} z \leq 3\pi/2$

particular value of arg

Leads to a new, single-valued Log etc.

Consider an interval o.t.f.

of the form

$\alpha \leq \tilde{\theta} < \alpha + 2\pi$ OR $\alpha < \tilde{\theta} \leq \alpha + 2\pi$

for some $\alpha \in \mathbb{R}$. We can define a single-valued argument on $\mathbb{C}_*$ with values in that interval, & hence a single-valued logarithm, called a branch of the logarithm. (*)

This then leads to being able to define a single-valued branch of, e.g., $z \mapsto z^{1/2}$.

(*) branch cut $\{z : \arg z = \alpha\} \cup \{0\}$.

E.g. $PV(z^{1/2})$ $\quad z \mapsto |z|^{1/2} \exp\left(\frac{i \operatorname{Arg} z}{2}\right)$

$re^{i\theta} \mapsto \sqrt{r}\exp(i\theta/2) \quad -\pi < \theta \leq \pi$

Branch cut : $-$ve Re axis $\cup \{0\}$.