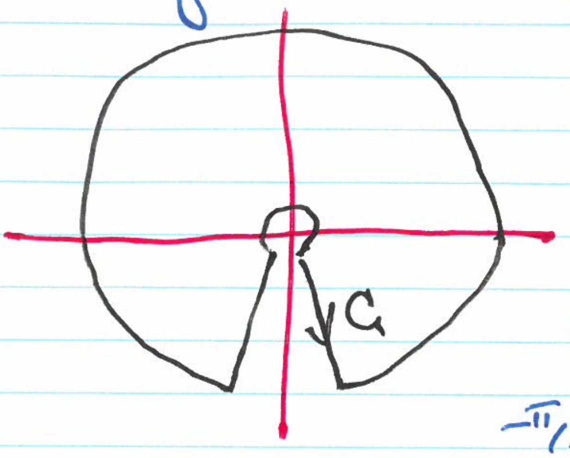
LECTURE 9

RMK: sometimes need to use a different single-valued Log & arg, e.g. integrating around a contour such as



C, called a keyhole contour. In this case, we could choose/define Arg s.t.

$$-\pi/2 < \underbrace{\mathrm{Arg}\, z}_{\text{particular value of arg } z} \le 3\pi/2,$$

which leads to a new, single-valued Log.

Consider any interval o.t.f. (of the form)

$$\alpha \le \tilde{\theta} < \alpha + 2\pi \quad \text{or} \quad \alpha < \tilde{\theta} \le \alpha + 2\pi,$$

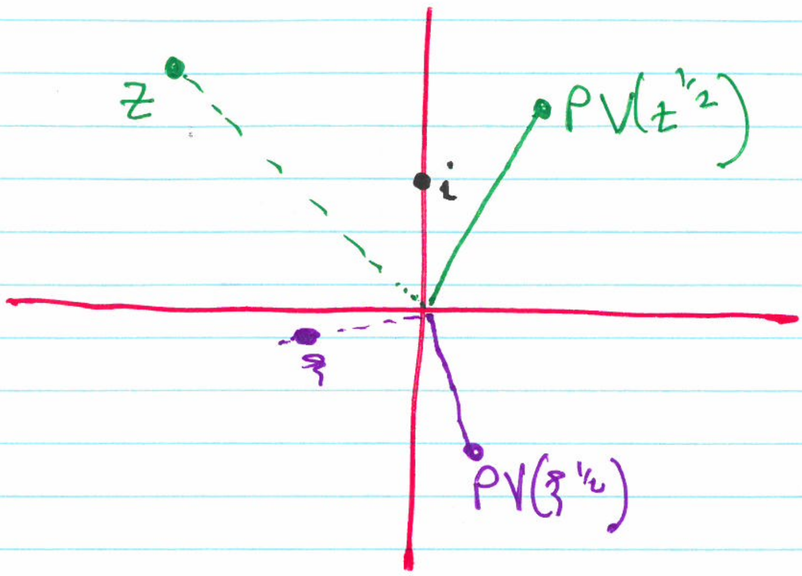
for some fixed $\alpha \in \mathbb{R}$. We can define a single-valued argument on $\mathbb{C}_*$, with values in that interval, & hence a single-valued logarithm, called a <u>branch of the logarithm</u>.

This leads to being able to define a single-valued branch of, e.g., $z \mapsto z^{1/2}$.

(*) <u>branch cut</u> : $\{x : \arg z = \alpha\} \cup \{0\}$.

E.g. $PV(z^{1/2})$ $\quad z \mapsto |z|^{1/2} \exp\left(\frac{i \operatorname{Arg} z}{2}\right)$

$re^{i\theta} \mapsto \sqrt{r}\, e^{i\theta/2} \qquad -\pi < \theta \le \pi.$



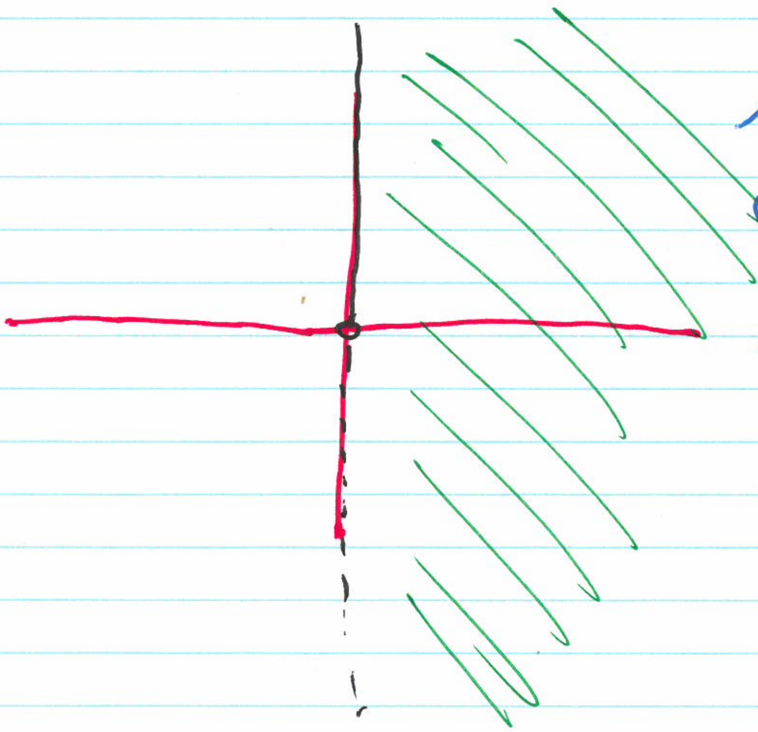
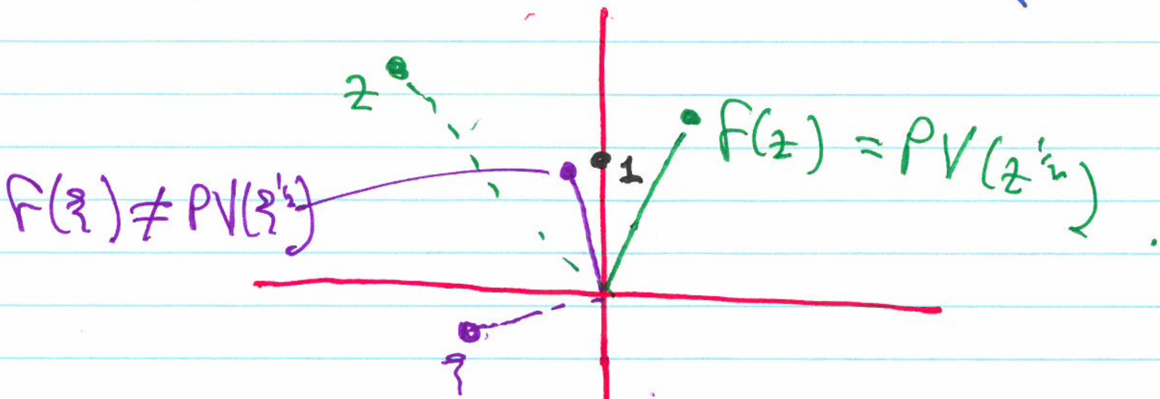
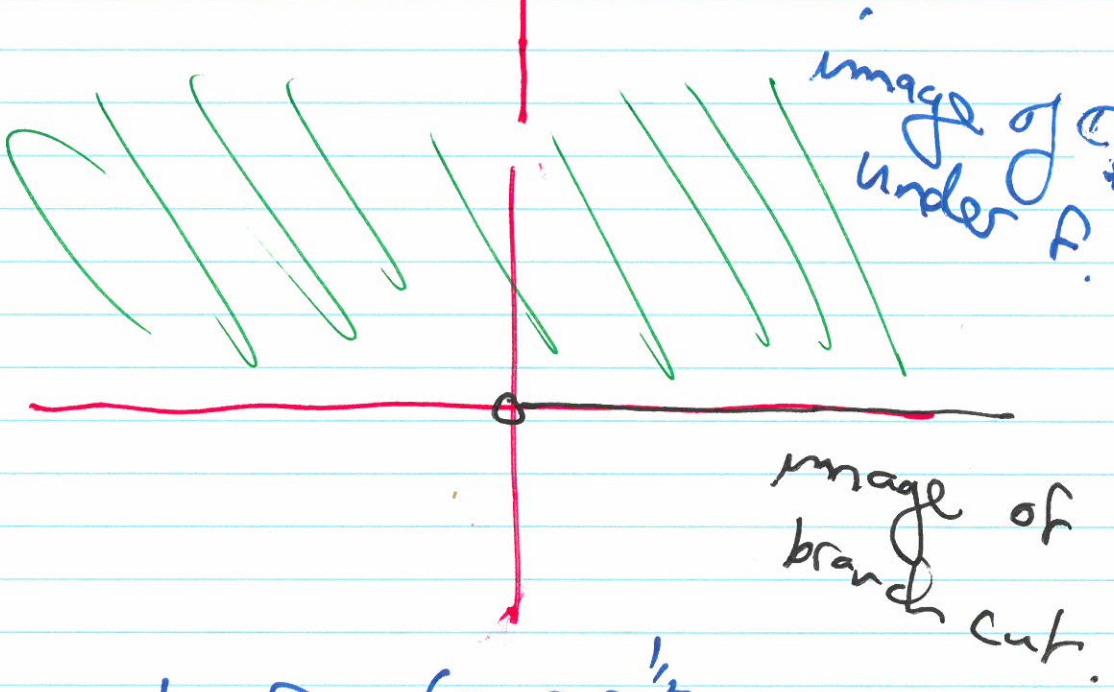


image of $\mathbb{C}^*$ under $PV(z^{1/2})$.

A different branch of $z^{1/2}$

$f : re^{i\theta} \mapsto \sqrt{r}\, e^{i\theta/2}$ $\quad 0 \le \theta < 2\pi$





repeat for $(z-z_0)^{1/2}$, $z_0 \in \mathbb{C}_*$.

§ 37–39 (8 Ed § 34–35) Trig.

$$e^{ix} = \cos x + i\sin x \quad (1) \qquad x \in \mathbb{R}$$

$$e^{-ix} = \cos x - i\sin x \quad (2)$$

(1) + (2) $\qquad \cos x = \dfrac{e^{ix} + e^{-ix}}{2} \quad (3)'$

(1) − (2) $\qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i} \quad (4)'$

Use (3)' & (4)' to define cos & sin on $\mathbb{C}$,

i.e. $\boxed{\cos z = \dfrac{e^{iz} + e^{-iz}}{2} \quad \& \quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}.}$

There hold $\qquad \cos z = \cos(-z) \quad (3)$

$$\sin z = -\sin(-z) \quad (4)$$

$$\cos(z+\zeta) = \cos z\cos\zeta - \sin z\sin\zeta; \quad (5)$$

$$\sin(z+\zeta) = \sin z\cos\zeta + \cos z\sin\zeta \quad (6)$$

$$\sin^2 z + \cos^2 z = 1 \quad (7)$$

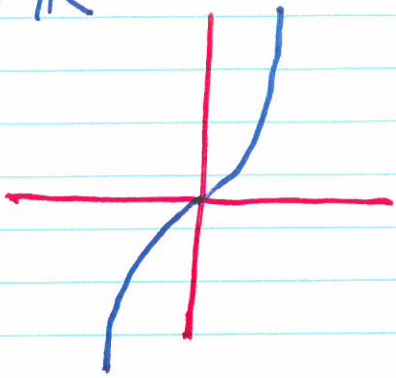
$$\sin(z + \pi/2) = \cos z \quad (8)$$

$$\sin(z - \pi/2) = -\cos z \quad (9)$$

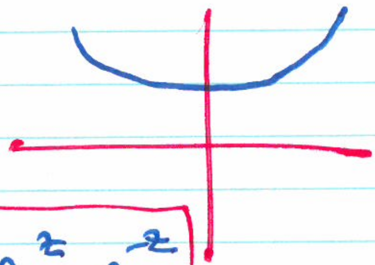
<u>Proofs</u>: use properties of exp, & previous results.

Recall: hyperbolic f^n's in $\mathbb{R}$

$$y=\sinh x = \frac{e^x - e^{-x}}{2}$$



$$y=\cosh x = \frac{e^x + e^{-x}}{2}$$



on $\mathbb{C}$, define

$$\boxed{\begin{aligned} \sinh z &= \frac{e^z - e^{-z}}{2} \\ \cosh z &= \frac{e^z + e^{-z}}{2} \end{aligned}}$$

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So: $\sin(iy) = \dfrac{e^{-y} - e^{y}}{2i} = \dfrac{i(e^{y} - e^{-y})}{2} = i\sinh y$ (10)

& $\cos(iy) = \dfrac{e^{-y} + e^{y}}{2} = \cosh y$. (11)