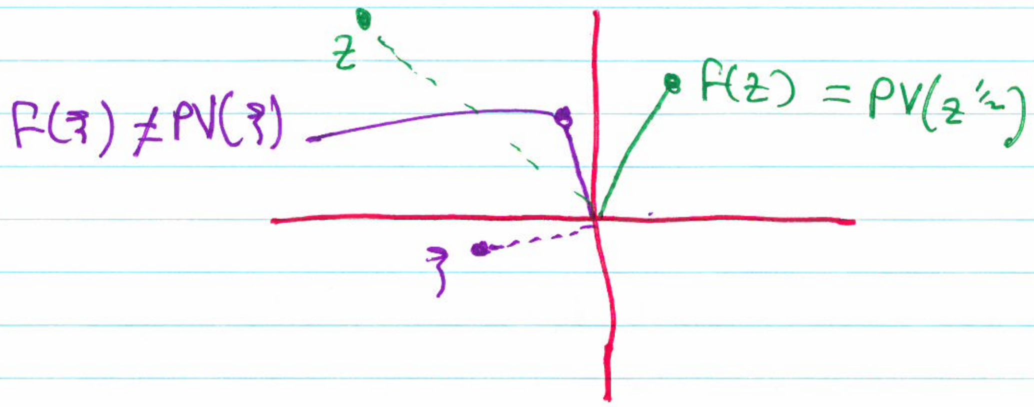
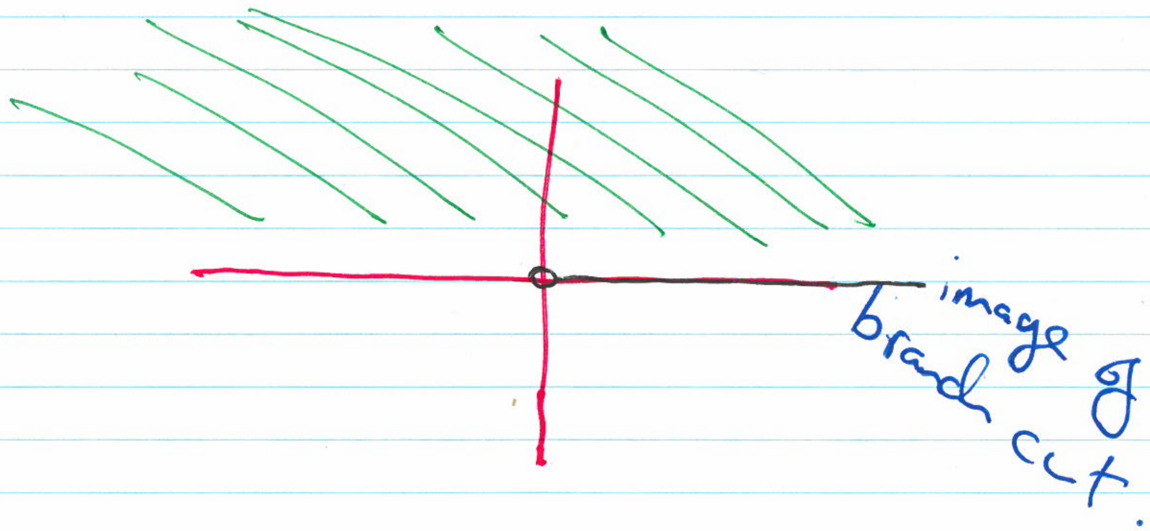
LECTURE 10

A different branch of $z^{1/2}$

$f : re^{i\theta} \mapsto \sqrt{r}\, e^{i\theta/2} \qquad 0 \leq \theta < 2\pi$





repeat for $(z-z_0)^{1/2}$, $z_0 \neq 0$ ☆

§37-39 (8 Ed §34-35) Trig

$e^{ix} = \cos x + i\sin x$ ① $\quad x \in \mathbb{R}$

$e^{-ix} = \cos x - i\sin x$ ②

① + ② $\Rightarrow$ $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ ③′

① − ② $\quad \sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$ ④′

Use ③′ & ④′ to define cos & sin on $\mathbb{C}$, i.e.,

$$\boxed{\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \& \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}}$$

These hold: $\cos z = \cos(-z)$; ③

$\sin z = -\sin(-z)$. ④

$\cos(z+\xi) = \cos z \cos\xi - \sin z \sin\xi$; ⑤

$\sin(z+\xi) = \sin z \cos\xi + \cos z \sin\xi$. ⑥

$\sin^2 z + \cos^2 z = 1$ ⑦

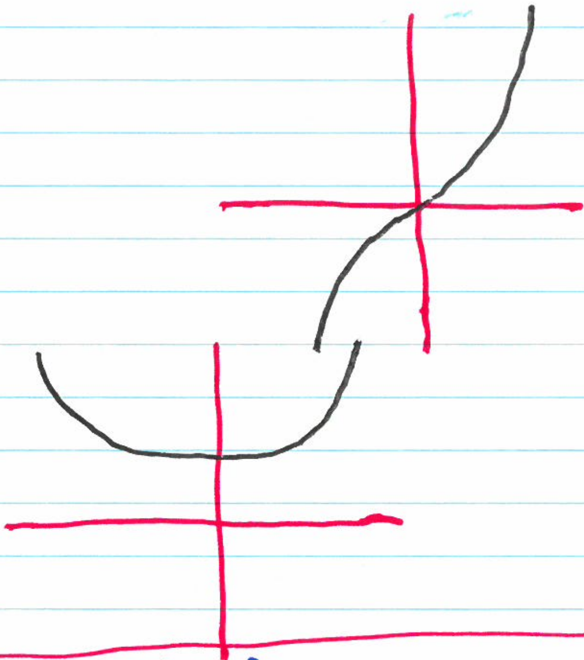
$\sin(z+\pi/2) = \cos z$ ⑧

$\sin(z-\pi/2) = -\cos z$ ⑨

Proofs: use properties of exp, & previous results. ☆

Recall: hyperbolic f^n's in $\mathbb{R}$

$y = \sinh x = \dfrac{e^{x} - e^{-x}}{2}$



$y = \cosh x = \dfrac{e^{x} + e^{-x}}{2}$

on $\mathbb{C}$, define $\boxed{\sinh z = \dfrac{e^{z} - e^{-z}}{2} \ \& \ \cosh z = \dfrac{e^{z} + e^{-z}}{2}}$

So: $\sin(iy) = \dfrac{e^{-y} - e^{y}}{2i} = \dfrac{i(e^{y} - e^{-y})}{2} = i \sinh y$ (10)

& $\cos(iy) = \dfrac{e^{-y} + e^{y}}{2} = \cosh y.$ (11)

Take $z = x$ & $\xi = iy$ in (5), (6):

$\sin(x+iy) \overset{(6)}{=} \sin x \cos(iy) + \cos x \sin(iy)$

$\overset{(10),(11)}{=} \sin x \cosh y + i \cos x \sinh y.$ (12)

Similarly:

$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$ (13) ☆

(12) & (13) $\Rightarrow$ $\sin(z + 2\pi) = \sin z$; (14)

$\cos(z+2\pi) = \cos z.$ (15)

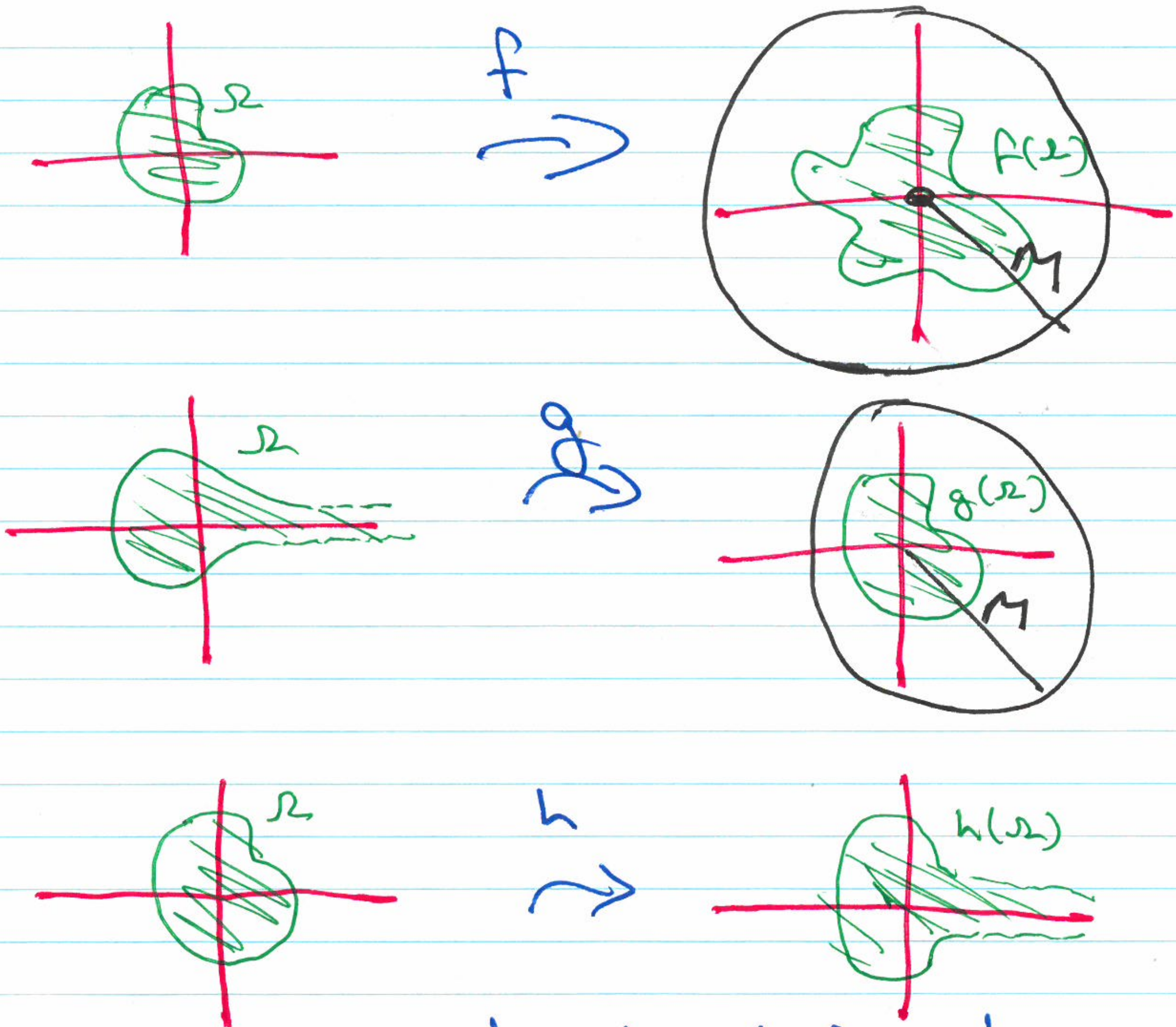
Note: $\cosh^2 z = 1 + \sinh^2 z$ ☆

$z = x+iy$:

$$|\sin z|^2 \overset{(12)}{=} \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$= \sin^2 x (1+\sinh^2 y) + \cos^2 x \sinh^2 y$$
$$= \sin^2 x (1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y.$$

Similarly: $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

__Def__: $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$, is __bounded__ if $\exists M \in \mathbb{R}$: $|f(z)| \leq M \ \forall z \in \Omega$.



f & g are bounded, h is not.

From the above, sin & cos are unbounded on $\mathbb{C}$.

bounded

* $f(z) = \pi$ is bded on $\mathbb{C}$;

* $f(z) = 1/z$ is unbded on $\mathbb{C}_*$;

" " " unbded on $\{z : 0 < |z| \leq 1\}$

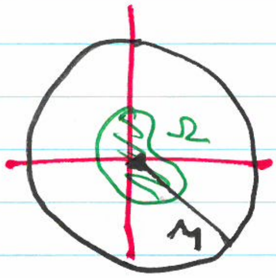
" " " bded on $\{z : |z| \geq 1\}$.

" " " bded on $\{z : |z| = 1\}$.

* $f(z) = z$ is unbded on $\mathbb{C}$, bded on any bded subset of $\mathbb{C}$.

<u>RMK:</u> Ω is a bounded subset of $\mathbb{C}$ says $\exists M \in \mathbb{R}$ s.t. $|z| \leq M \ \forall z \in \Omega$.

* $f(z) = \dfrac{1}{1+|z|}$ is bded on $\mathbb{C}$.



<u>Def:</u> a zero of a f^n is a value of z s.t. $f(z) = 0$.

E.g. zeros of sin : (12) $\Rightarrow n\pi + 0i \quad n \in \mathbb{Z}$;

zeros of cos (13) $\Rightarrow (n+\tfrac{1}{2})\pi + 0i \quad n \in \mathbb{Z}$.

These are the only zeros of sin/cos.

Similarly:

zeros of sinh : $n\pi i \quad n \in \mathbb{Z}$;

zeros of cosh : $(n+\tfrac{1}{2})\pi i \quad n \in \mathbb{Z}$.

These are their only zeros.

§40 (8 Ed §37) $\sin^{-1}$, $\cos^{-1}$

"$w = \sin^{-1} z$"

$$z = \sin w$$

$$= \frac{e^{iw} - e^{-iw}}{2i} \cdot \frac{e^{iw}}{e^{iw}}$$

$$= \frac{e^{2iw} - 1}{2ie^{iw}}$$

$$\Rightarrow \quad 2ize^{iw} = e^{2iw} - 1.$$

Put $\zeta = e^{iw}$

$$\Rightarrow \quad 2iz\zeta = \zeta^2 - 1$$

$$\Rightarrow \quad \zeta^2 - 2iz\zeta - 1 = 0.$$

in general, double valued

$$\Rightarrow \quad \zeta = e^{iw} = \frac{2iz + (-4z^2 + 4)^{1/2}}{2}$$

$$= iz + (1 - z^2)^{1/2}$$

$$\Rightarrow \quad iw = \log\left(iz + (1 - z^2)^{1/2}\right)$$

multi valued

$$\Rightarrow \quad w = -i\log\left(iz + \underbrace{(1 - z^2)^{1/2}}_{\text{double-valued}}\right)$$

multi-valued