LECTURE 10

Indeed, for $z \in \mathbb{C}$, $\sin iz = i \sinh z$ (10)'

& $\cos iz = \cosh z$ (11)'

Moreover $-i\sinh(iz) = \sin z$ (12)

& $\cosh iz = \cos z$ (13)

In analogue to (3) & (4), we have

$$\sinh(-z) = -\sinh z \quad (14)$$

& $$\cosh(-z) = \cosh z \quad (15)$$

In analogue to (5) & (6), we have

$$\sinh(z+\xi) = \sinh z \cosh \xi + \cosh z \sinh \xi \quad (16)$$

& $$\cosh(z+\xi) = \cosh z \cosh \xi + \sinh z \sinh \xi \quad (17)$$

Taking $z = x$ & $\xi = iy$ in (6)

$$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

$$\overset{(10),(11)}{=} \sin x \cosh y + i \cos x \sinh y. \quad (18)$$

Similarly, via (5), (10) & (11) ✱

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y \quad (19)$$

& via (10), (11), (16) & (17) ✱

$$\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y, \quad (20)$$

$$\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y \quad (21)$$

Note, cf. (7) (i.e. $\sin^2 z + \cos^2 z = 1$):

$$\cosh^2 z = 1 + \sinh^2 z. \quad (22)$$

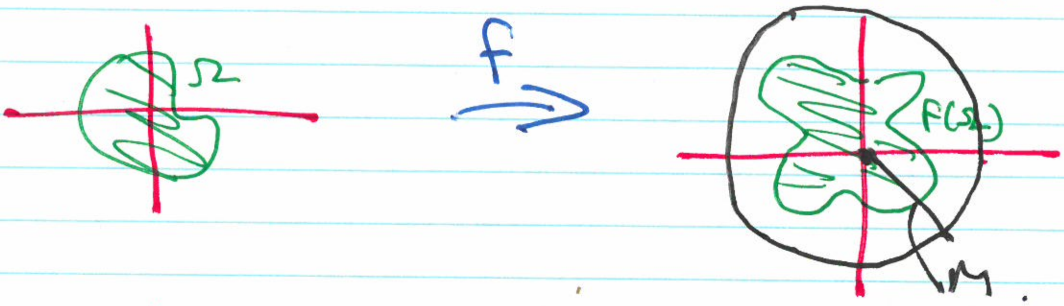
Note also for $z = x + iy$:

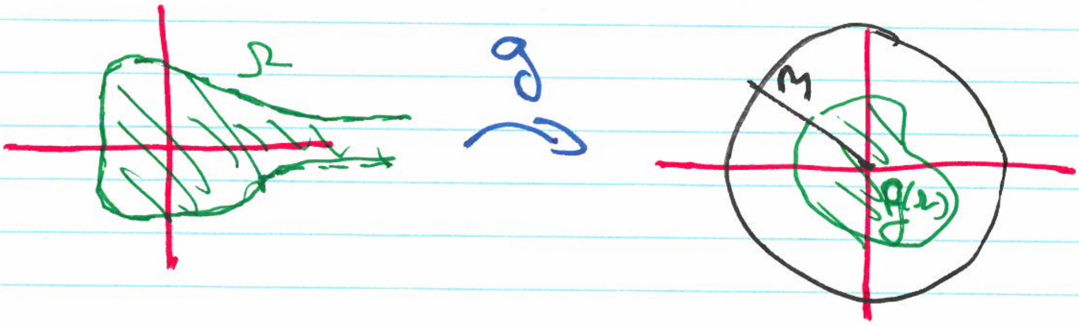
$$|\sin z|^2 \overset{(18)}{=} \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$\overset{(22),(7)}{=} \sin^2 x (1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y \qquad (23)$$

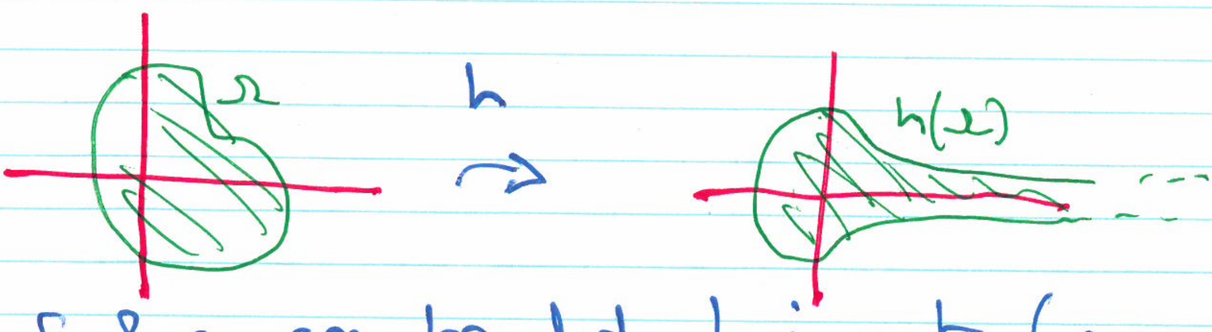
Similarly from (19), (20) & (21) using (7) & (22)

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \qquad (24)$$
$$|\sinh z|^2 = \sinh^2 x + \sin^2 y. \qquad (25)$$
$$|\cosh z|^2 = \sinh^2 x + \cos^2 y. \qquad (26)$$

Defn: $f : \Omega \to \mathbb{C}$ is **bounded** if $\exists M \in \mathbb{R}$:
$\Omega \subseteq \mathbb{C}$ $\quad |f(z)| \leq M \;\; \forall z \in \Omega$.







f & g are bounded, h is not (so, h is **unbounded**).

RMK: Ω is a bounded subset of $\mathbb{C}$ says $\exists\ M \in \mathbb{R}$ s.t. $|z| \leq M\ \forall z \in \Omega$.

* $f(z) = \pi$ is bded on $\mathbb{C}$;
  bounded.
* $f(z) = 1/z$ is unbded on $\mathbb{C}_*$;
  " " " " on $\{z : 0 < |z| \leq 1\}$
  " " is bded on $\{z : |z| \geq 1\}$
  " " is bded on $\{z : |z| = 1\}$
* $f(z) = z$ is unbded on $\mathbb{C}$, bded on any bded subset of $\mathbb{C}$.
* From (23), (24), sin & cos are bounded on $\mathbb{R}$, unbounded on $\mathbb{C}$;
* From (25), (26) sinh & cosh are unbded on $\mathbb{R}$ & on $\mathbb{C}$.
* $f(z) = 1/_{1+|z|}$ is bded on $\mathbb{C}$.

Def^n: a zero of f^n f is a value z s.t. $f(z) = 0$.

E.g. zeros of sin (18) $\Rightarrow$ $n\pi + 0i$ $n \in \mathbb{Z}$;
zeros of cos (19) $\Rightarrow (n+\frac{1}{2})\pi + 0i$ $n \in \mathbb{Z}$;
these are the only zeros of sin/cos.

Similarly via (20), (21)
zeros of sinh $n\pi i$ $n \in \mathbb{Z}$
zeros of cosh $(n+\frac{1}{2})\pi i$ $n \in \mathbb{Z}$
These are the only zeros of sinh/cosh.

§40 (8 Ed §37) $\sin^{-1}$, $\cos^{-1}$

" $w = \sin^{-1} z$ "

$$z = \sin w$$

$$= \frac{e^{iw} - e^{-iw}}{2i} \cdot \frac{e^{iw}}{e^{iw}}$$

$$= \frac{e^{2iw} - 1}{2ie^{iw}}$$

$$\Rightarrow \quad 2ize^{iw} = e^{2iw} - 1.$$

Put $q = e^{iw}$

$$\Rightarrow 2izq = q^2 - 1.$$

$$\Rightarrow q^2 - 2izq - 1 = 0.$$

$$\Rightarrow q = e^{iw} = \frac{2iz + (-4z^2 + 4)^{1/2}}{2}$$

in general, double-valued

$$= iz + (1 - z^2)^{1/2}$$

$$\Rightarrow \quad iw = \underbrace{\mathrm{Log}}_{\text{multi-valued}}\left(iz + (1 - z^2)^{1/2}\right)$$

$$\Rightarrow \quad w = \sin^{-1} z = -i\underbrace{\mathrm{Log}}_{\text{multi-valued}}\left(iz + \underbrace{(1 - z^2)^{1/2}}_{\text{double-valued}}\right)$$