LECTURE 10

§37-39 (8 Ed §34-35) Trig.

$$e^{ix} = \cos x + i\sin x \quad ① \qquad x \in \mathbb{R}.$$

$$e^{-ix} = \cos x - i\sin x \quad ②$$

① + ②
$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \quad ③'$$

① − ②
$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} \quad ④'$$

Use ③' & ④' to define sin & cos on $\mathbb{C}$, i.e.:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \&$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

These hold
$$\cos z = \cos(-z) \quad ③$$
$$\sin z = -\sin(-z) \quad ④$$

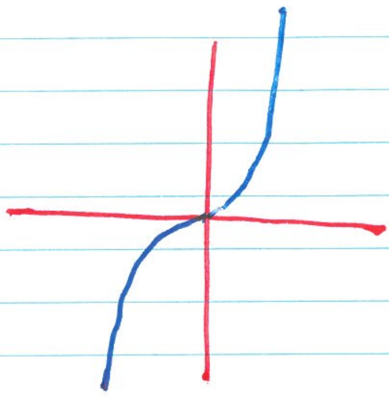
$$\cos(z+\xi) = \cos z\cos\xi - \sin z\sin\xi \quad ⑤$$
$$\sin(z+\xi) = \sin z\cos\xi + \cos z\sin\xi \quad ⑥$$
$$\sin^2 z + \cos^2 z = 1 \quad ⑦$$
$$\sin(z+\pi/2) = \cos z \quad ⑧$$
$$\sin(z-\pi/2) = -\cos z \quad ⑨$$

Proofs: use properties of exp. & previous results.

Recall: hyperbolic f^n-s in $\mathbb{R}$:

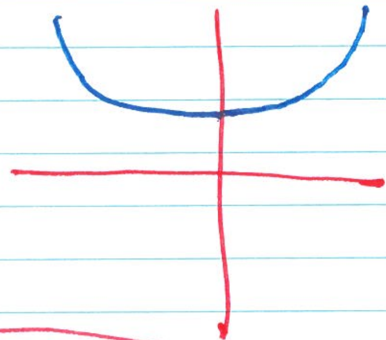
$$y = \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$x \in \mathbb{R}$



$$y = \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$x \in \mathbb{R}$



on $\mathbb{C}$, define

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

& $$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\sin(iz) = \frac{e^{-z} - e^{z}}{2i} = \frac{i(e^{z} - e^{-z})}{2} = i \sinh z \quad (10)$$

& $$\cos(iz) = \frac{e^{-z} + e^{z}}{2} = \cosh z \quad (11)$$

Moreover: $-i \sinh(iz) = \sin z$ (12)

& $\cosh iz = \cos z$ (13)

In analogue to (3) & (4), we have

$$\sinh(-z) = -\sinh z \quad \& \quad (14)$$

$$\cosh(-z) = \cosh z. \quad (15)$$

In analogue to (5) & (6), we have

$$\sinh(z+\xi) = \sinh z \cosh\xi + \cosh z \sinh\xi \quad (16)$$

$$\& \cosh(z+\xi) = \cosh z \cosh\xi + \sinh z \sinh\xi \quad (17)$$

Take $z = x$ & $\xi = iy$ in (6)

$$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

$$\overset{(10),(11)}{=} \sin x \cosh y + i \cos x \sinh y. \quad (18)$$

Similarly, via (5), (10) & (11) ✱

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y \quad (19)$$

&, via (10), (11), (16) & (17) ✱

$$\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y \quad (20)$$

$$\& \cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y \quad (21)$$

Note, c.f. (7) (i.e. $\sin^2 z + \cos^2 z = 1$):

$$\cosh^2 z = 1 + \sinh^2 z. \quad (22)$$

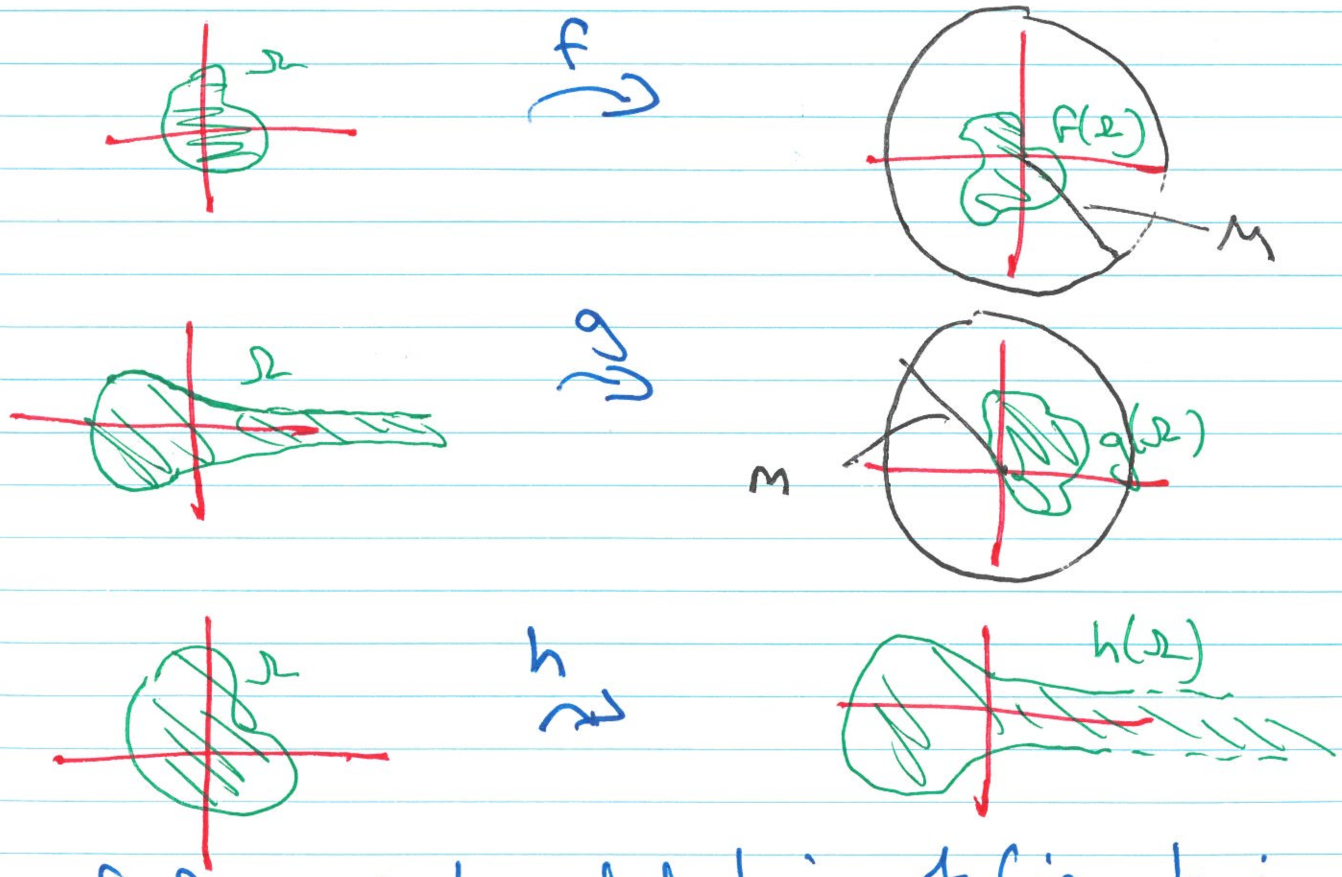
Note also for $z = x + iy$:

$$|\sin z|^2 \overset{(18)}{=} \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$\overset{(22),(7)}{=} \sin^2 x (1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y \quad (23)$$

Similarly from (19), (20) & (21) using (7) & (22)

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad (24)$$
$$|\sinh z|^2 = \sinh^2 x + \sin^2 y \quad (25)$$
$$|\cosh z|^2 = \sinh^2 x + \cos^2 y \quad (26)$$

__Def^n:__ $f : \Omega \to \mathbb{C}$ (with $\Omega \subseteq \mathbb{C}$) is bounded if $\exists M \in \mathbb{R}$: $|f(z)| \leq M \ \forall z \in \Omega$.



f & g are bounded, h is not (i.e., h is <u>unbounded</u>).

RMK: Ω is a bounded subset of $\mathbb{C}$ says $\exists M \in \mathbb{R}$ s.t. $|z| \leq M$ $\forall z \in \Omega$.

* $f(z) = \pi$ is bded (bounded) on $\mathbb{C}$;
* $f(z) = 1/z$ is unbded on $\mathbb{C}_*$;
  " " " " on $\{z : 0 < |z| \leq 1\}$
  " " " bded on $\{z : |z| \geq 1\}$
  " " " bded on $\{z : |z| = 1\}$.
* $f(z) = z$ is unbded on $\mathbb{C}$ & any unbded subset of $\mathbb{C}$, bded on any bded subset of $\mathbb{C}$:
* From (23) & (24), sin & cos are bded on $\mathbb{R}$, unbded on $\mathbb{C}$;
* From (25) & (26), sinh & cosh are unbded on $\mathbb{R}$ & on $\mathbb{C}$.
* $f(z) = \frac{1}{1+|z|}$ is bded on $\mathbb{C}$.