LECTURE 11

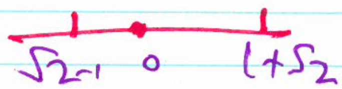
Recall $\sin^{-1}(z) = -i\,\mathrm{Log}\left(iz + (1-z^2)^{1/2}\right)$

E.g. $\sin^{-1}(-i) = -i\log\left(1 + 2^{1/2}\right)$ (∗)

$$2^{1/2} = \left(2e^{i0}\right)^{1/2}$$
$$= \left\{\sqrt{2}e^{i\cdot 0/2},\ \sqrt{2}e^{i(0+2\pi)/2}\right\}$$
$$= \left\{\sqrt{2},\ \sqrt{2}e^{i\pi}\right\}$$
$$= \left\{\pm\sqrt{2}\right\}.$$

So, note $\log(1+\sqrt{2}) = \ln(1+\sqrt{2}) + \overbrace{2n\pi}^{\arg(1+\sqrt{2})} i,\ n \in \mathbb{Z}$

$\log(1-\sqrt{2}) = \ln \underbrace{(\sqrt{2}-1)}_{=|1-\sqrt{2}|} + \underbrace{(2n+1)\pi}_{\arg(1-\sqrt{2})} i \quad n \in \mathbb{Z}.$



$$\Rightarrow \sin^{-1}(-i) = \left\{-i\left[\ln(1+\sqrt{2}) + 2n\pi i\right],\ n \in \mathbb{Z}\right\} \cup$$
$$\left\{-i\left[\ln(\sqrt{2}-1) + (2n+1)\pi i\right],\ n \in \mathbb{Z}\right\}.$$
$$= \left\{-i\ln(1+\sqrt{2}) + 2n\pi,\ n \in \mathbb{Z}\right\} \cup$$
$$\left\{-i\ln(\sqrt{2}-1) + (2n+1)\pi,\ n \in \mathbb{Z}\right\}.$$

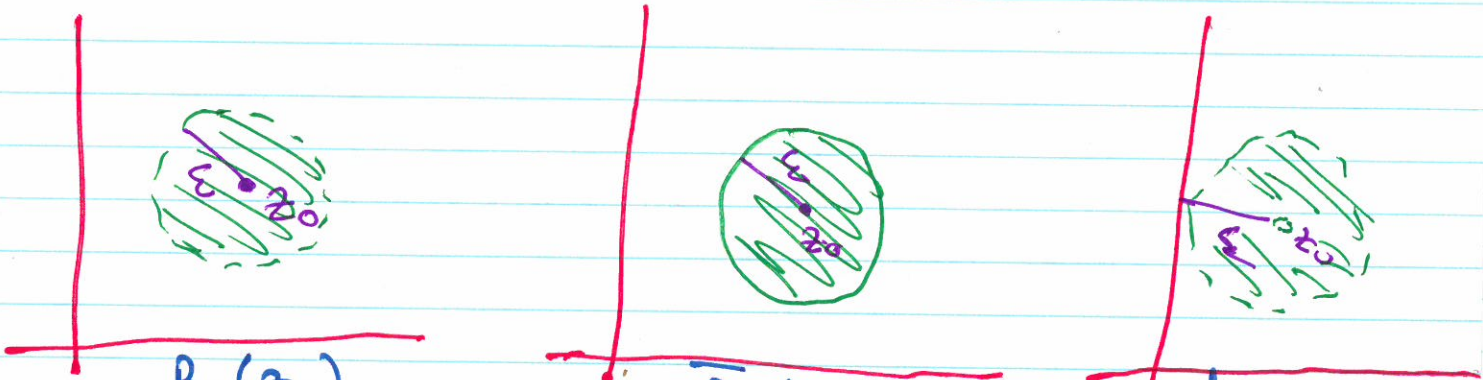
§12 (§11 8Ed) Topology

* Given $z_0 \in \mathbb{C}$ & $\varepsilon > 0$, $B_\varepsilon(z_0)$ denotes the (open) ball of radius ε about z_0, a.k.a. ε-neighbourhood of z_0 given by $\boxed{\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}}$

* $\overline{B}_\varepsilon(z_0)$ = closed ball of radius ε about z_0, a.k.a. closed ε-nbhd of z_0, given by

$$\boxed{z \in \mathbb{C} : |z - z_0| \leq \varepsilon.}$$

* Deleted ε-nbhd of z_0 : $\{z : 0 < |z - z_0| < \varepsilon\}$.



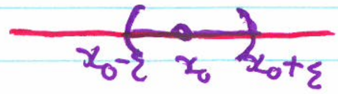
$B_\varepsilon(z_0)$ $\quad$ $\overline{B}_\varepsilon(z_0)$ $\quad$ deleted ε-nbhd of z_0

$$b_\varepsilon(z_0) = \{z : |z - z_0| < \varepsilon\}$$

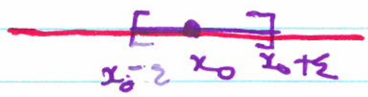
$$\begin{aligned}
|z - z_0| &= |(x + iy) - (x_0 + iy_0)| \\
&= |(x - x_0) + i(y - y_0)| \\
&= \sqrt{(x - x_0)^2 + (y - y_0)^2} \\
&= \|(x, y) - (x_0, y_0)\|_{\mathbb{R}^2} \\
&= d_{\mathbb{R}^2}((x, y), (x_0, y_0)) \\
&= d_{\mathbb{C}}(z, z_0)
\end{aligned}$$

$d(\cdot, \cdot)$ = distance.

In $\mathbb{R}$



$B_\varepsilon(x_0)$



$\overline{B}_\varepsilon(x_0)$

Take $\Omega \subseteq \mathbb{C}$:

* $z \in \mathbb{C}$ is an interior point of Ω if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(z) \subset \Omega$. (Note $\Rightarrow B_{\varepsilon'}(z) \subset \Omega \;\; \forall \varepsilon',\ 0 < \varepsilon' < \varepsilon$).

such that

* $z \in \mathbb{C}$ is an exterior point of Ω if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(z) \cap \Omega = \emptyset$ — empty set.

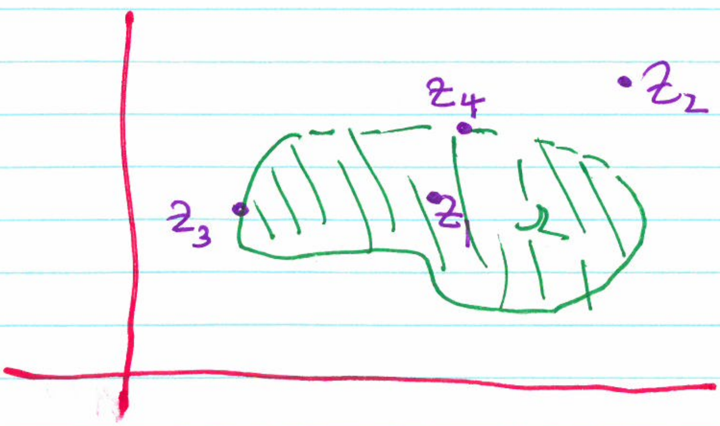
* $z \in \mathbb{C}$ is a boundary point of Ω, $z \in \partial\Omega$ if $\forall\, \varepsilon > 0$ there holds: $B_\varepsilon(z) \cap \Omega \neq \emptyset$ AND

$B_\varepsilon(z) \cap \Omega^c \neq \emptyset$

complement of Ω, i.e., $\mathbb{C} \setminus \Omega$.

$\partial\Omega$ = boundary of $\Omega = \{ z \in \mathbb{C} : z$ is a boundary pt of $\Omega \}$.

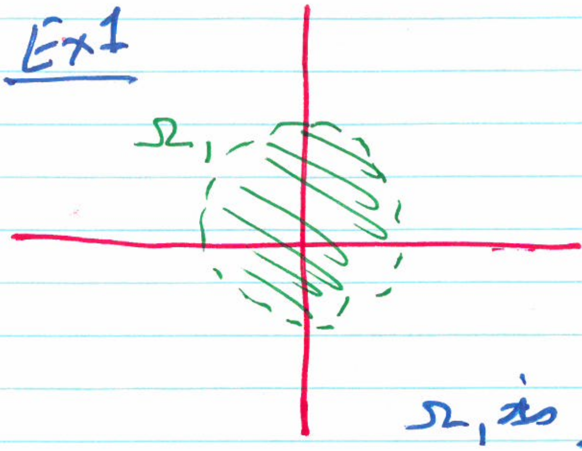
Note: Interior points belong to Ω;
Exterior points belong to Ω^c;
Bdy pts: ??



z_1: interior point
z_2: exterior pt
z_3: bdy pt, $z_3 \in \Omega$
z_4: bdy pt, $z_4 \notin \Omega$.

* $\operatorname{Int} \Omega$ = interior of Ω :
  $= \{z : z \text{ is an interior pt of } \Omega\}$
* $\operatorname{Ext} \Omega$ = exterior of Ω
  $= \{z : z \text{ is an exterior pt of } \Omega\}$.
* Ω is open if $\Omega = \operatorname{Int} \Omega$.
* Ω is closed if $\partial\Omega \subseteq \Omega$.

Ex 1

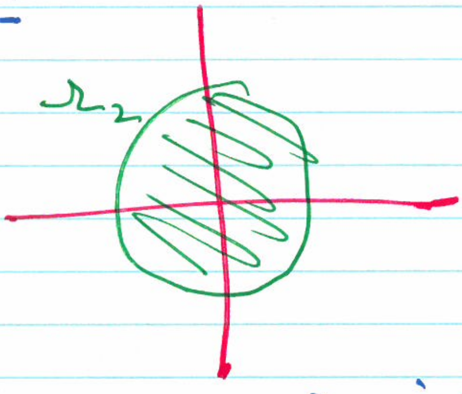


$\Omega_1 = B_1(0) = B_1 = B = \{z : |z| < 1\}$

- $\operatorname{Int} \Omega_1 = \Omega_1$
- $\operatorname{Ext} \Omega_1 = \{z : |z| > 1\}$
- $\partial \Omega_1 = \{z : |z| = 1\} = S^1$
- $\Omega_1^c = \{z : |z| \geq 1\}$.

Ω_1 is open.

Ex 2

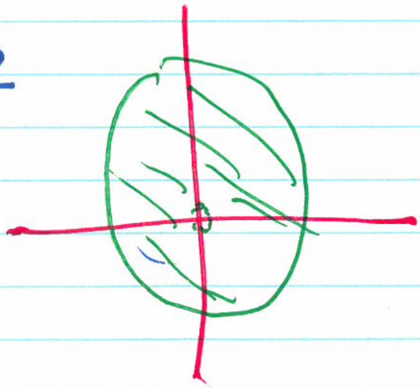


$\Omega_2 = \overline{B}_1(0) = \overline{B}_1 = \overline{B} = \{z : |z| \leq 1\}$

- $\operatorname{Int} \Omega_2 = \Omega_1$
- $\operatorname{Ext} \Omega_2 = \operatorname{Ext} \Omega_1$
- $\partial \Omega_2 = S^1 = \partial \Omega_1$
- $\Omega_2^c = \operatorname{Ext} \Omega_2 = \operatorname{Ext} \Omega_1$

Ω_2 is closed.

Ex 3



$\Omega_3 = \{z : 0 < |z| \leq 1\}$

- $\operatorname{Int} \Omega_3 = \{z : 0 < |z| < 1\}$
- $\operatorname{Ext} \Omega_3 = \operatorname{Ext} \Omega_1$
- $\partial \Omega_3 = S^1 \cup \{0\}$
- $\Omega_3^c = \operatorname{Ext} \Omega_1 \cup \{0\}$.

Ω_3 is neither open nor closed.