LECTURE 11

Def^n: a zero of a f^n: f is a value z s.t. $f(z) = 0$.

E.g. zeros of sin (18) $\Rightarrow n\pi + 0i \quad n \in \mathbb{Z}$;

zeros of cos (19) $\Rightarrow (n+\frac{1}{2})\pi + 0i \quad n \in \mathbb{Z}$.

These are the only zeros in $\mathbb{C}$ of sin/cos.

Similarly, (20) & (21) $\Rightarrow$

zeros of sinh are $n\pi i \quad n \in \mathbb{Z}$;

zeros of cosh are $(n+\frac{1}{2})\pi i \quad n \in \mathbb{Z}$.

These are the only zeros of sinh/cosh.

§40 (8Ed §37) $\sin^{-1}$, $\cos^{-1}$

"$w = \sin^{-1} z$" $\Rightarrow$ $z = \sin w$

$$= \frac{e^{iw} - e^{-iw}}{2i} \cdot \frac{e^{iw}}{e^{iw}}$$

$$= \frac{e^{2iw} - 1}{2ie^{iw}}$$

$$\Rightarrow 2ize^{iw} = e^{2iw} - 1.$$

Put $\zeta = e^{iw}$

$$\Rightarrow 2iz\zeta = \zeta^2 - 1.$$

$$\Rightarrow \zeta^2 - 2iz\zeta - 1 = 0.$$

$$\Rightarrow \zeta = e^{iw} = \frac{2iz + (-4z^2 + 4)^{1/2}}{2}$$

in general, double-valued

$$= iz + (1 - z^2)^{1/2}$$

$$iw = \underbrace{\log}_{\text{multi-valued.}}(iz + (1 - z^2)^{1/2})$$

$$\Rightarrow \quad w = \sin^{-1} z = -i \underbrace{\log}_{\text{multi-valued}} (iz + \underbrace{(1 - z^2)^{1/2}}_{\text{double-valued.}})$$

E.g., $\sin^{-1}(-i) = -i\log(1+2^{1/2})$ *

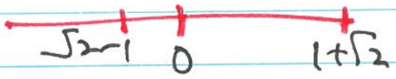
$$2^{1/2} = \left(2e^{i\cdot 0}\right)^{1/2}$$
$$= \left\{\sqrt{2}\,e^{i0/2},\ \sqrt{2}\,e^{i(0+2\pi)/2}\right\}$$
$$= \left\{\sqrt{2},\ \sqrt{2}\,e^{i\pi}\right\}$$
$$= \pm\sqrt{2}.$$

$\arg(1+\sqrt{2})$ "

So, note $\log(1+\sqrt{2}) = \ln(1+\sqrt{2}) + 2k\pi i \qquad k\in\mathbb{Z}$

& $\log(1-\sqrt{2}) = \ln(\sqrt{2}-1) + (2k+1)\pi i \qquad k\in\mathbb{Z}.$

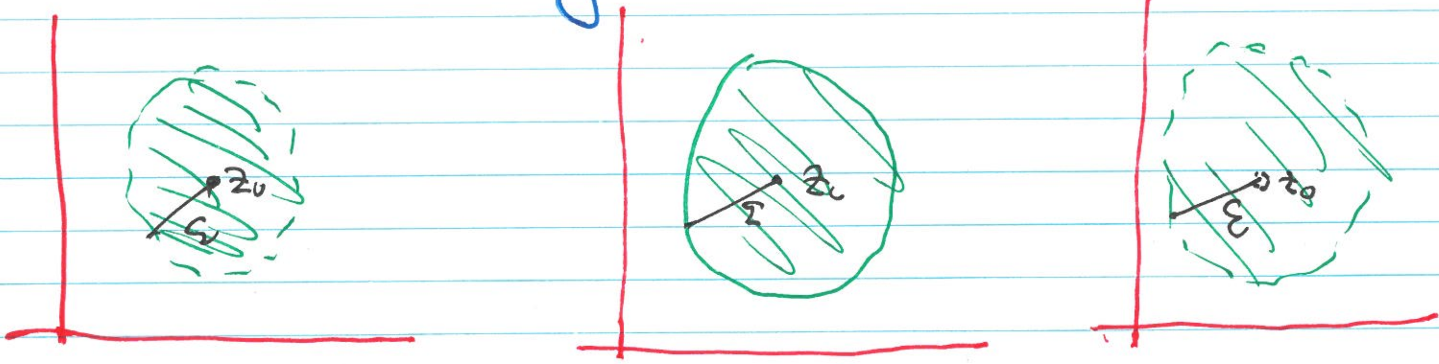
(where $\sqrt{2}-1 = |1-\sqrt{2}|$)



$$\Rightarrow \sin^{-1}(-i) = \left\{-i\left[\ln(1+\sqrt{2}) + 2k\pi i,\ k\in\mathbb{Z}\right]\right\} \cup$$
$$\left\{-i\left[\ln(\sqrt{2}-1) + (2k+1)\pi i,\ k\in\mathbb{Z}\right]\right\}$$
$$= \left\{2k\pi - i\ln(1+\sqrt{2}),\ k\in\mathbb{Z}\right\} \cup$$
$$\left\{(2k+1)\pi - i\ln(\sqrt{2}-1),\ k\in\mathbb{Z}\right\}.$$

§12 (8Ed §11) Topology

* Given $z_0 \in \mathbb{C}$ & $\varepsilon > 0$, $B_\varepsilon(z_0)$ denotes the (open) ball of radius ε about z_0, a.k.a. the ε-neighbourhood of z_0, given by $\boxed{\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}}$.
* $\overline{B}_\varepsilon(z_0)$ = closed ball of radius ε about z_0, a.k.a. closed ε-nbhd of z_0, given by $\boxed{\{z \in \mathbb{C} : |z - z_0| \le \varepsilon\}}$.
* Deleted ε-nbhd of z_0 : $\{z : 0 < |z - z_0| < \varepsilon\}$

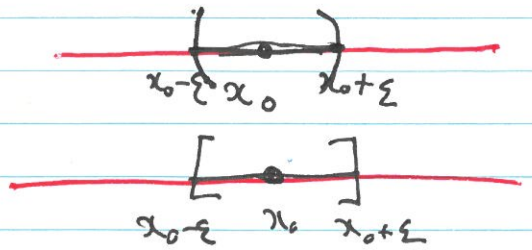


$$B_\varepsilon(z_0) = \{z : |z - z_0| < \varepsilon\}$$

$$\begin{aligned}
|z - z_0| &= |(x+iy) - (x_0 + iy_0)| \\
&= |(x - x_0) + i(y - y_0)| \\
&= \sqrt{(x - x_0)^2 + (y - y_0)^2} \\
&= \|(x,y) - (x_0, y_0)\|_{\mathbb{R}^2} \\
&= d\big((x,y), (x_0,y_0)\big)_{\mathbb{R}^2} \\
&= d(z, z_0)_{\mathbb{C}}
\end{aligned}$$

$d(\cdot,\cdot)$ = distance.

In $\mathbb{R}$



$B_\varepsilon(x_0)$

$\overline{B_\varepsilon}(x_0)$

Take $\Omega \subseteq \mathbb{C}$:

* $z \in \mathbb{C}$ is an interior point of Ω if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(z) \subseteq \Omega$ (note $\Rightarrow B_{\varepsilon'}(z) \subset \Omega \;\; \forall \varepsilon',\, 0<\varepsilon'<\varepsilon$; and $\Rightarrow z \in \Omega$).
* $z \in \mathbb{C}$ is an exterior point of Ω if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(z) \cap \Omega = \emptyset$.
* $z \in \mathbb{C}$ is a boundary point of Ω, $z \in \partial\Omega$, if $\forall \varepsilon > 0$ there holds

$$B_\varepsilon(z) \cap \Omega \neq \emptyset \;\; \& \;\; B_\varepsilon(z) \cap \Omega^c \neq \emptyset .$$

$\rightarrow$ complement of Ω, i.e., $\mathbb{C} \setminus \Omega$.

$\partial\Omega$ = boundary of Ω = $\{ z \in \mathbb{C} : z$ is a bdy pt of $\Omega \}$

Note: Interior pts belong to Ω;
Exterior pts belong to Ω^c;
Bdy pts: ??

* $\operatorname{Int} \Omega$ = interior of Ω $= \{ z : z$ is an interior pt of $\Omega \}$
* $\operatorname{Ext} \Omega$ = exterior of Ω $= \{ z : z$ is an exterior pt of $\Omega \}$.
* Ω is open if $\Omega = \operatorname{Int} \Omega$.
* Ω is closed if $\partial\Omega \subseteq \Omega$.