LECTURE 12

Note: Ω_1 (from Lecture 11) is open, Ω_1^c is closed;
Ω_2 " is closed, Ω_2^c is open.

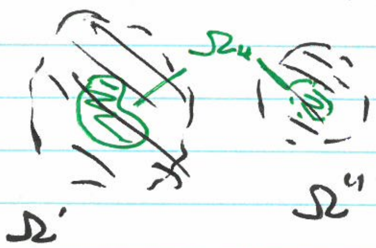
Neither Ω_3 nor Ω_3^c is open, nor is either closed.

Only clopen sets in $\mathbb{C}$ are $\emptyset$ & $\mathbb{C}$.

closed & open

* $\Omega \subseteq \mathbb{C}$ is connected if there do not exist non-empty, disjoint open sets Ω' & Ω'' s.t. $\Omega \subseteq \Omega' \cup \Omega''$, &

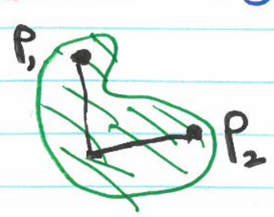
$$\Omega' \cap \Omega \neq \emptyset \ \& \ \Omega'' \cap \Omega \neq \emptyset.$$



Ω_4 is not connected, i.e., it is disconnected

Ω_5 is connected, as are Ω_1, Ω_2, Ω_3.

* $\Omega \subseteq \mathbb{C}$ is piecewise affinely path connected if any 2 points in Ω can be connected by a finite # of line segments in Ω, joined end-to-end.

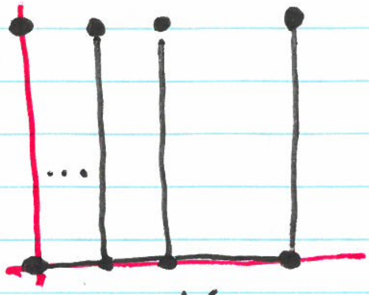


For open sets in $\mathbb{C}$, the two definitions are equivalent.

In general, the two def^n's are not equivalent.

Consider $K = \bigcup_{n=1}^{\infty} \{ \frac{1}{n} + yi, \ 0 \le y \le 1 \}$

$\cup \{ x + 0i, \ 0 \le x \le 1 \} \cup \{ i \}$.



K is connected, but is not p.w. aff. p.c., as you can't get a path from i to any other point in K that stays in K.

Example proof in this area:

If Ω_1 & Ω_2 are open in $\mathbb{C}$, then so is $\Omega_1 \cap \Omega_2$.

Pf: If $\Omega_1 \cap \Omega_2 = \emptyset$, we're done ($\emptyset$ is open).

If not, for any $z \in \Omega_1 \cap \Omega_2$

$z \in \Omega_1 \Rightarrow \exists \, \varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(z) \subset \Omega_1$, ⊛

& $z \in \Omega_2 \Rightarrow \exists \, \varepsilon_2 > 0$ s.t. $B_{\varepsilon_2}(z) \subset \Omega_2$ ☺,

since Ω_1 & Ω_2 are open.

So, set $\varepsilon = \min \{ \varepsilon_1, \varepsilon_2 \}$: note $\varepsilon > 0$.

minimum

$\Rightarrow B_\varepsilon(z) \subset \Omega_1$ by ⊛ & $B_\varepsilon(z) \subset \Omega_2$ by ☺.

Hence, $B_\varepsilon(z) \subset \Omega_1 \cap \Omega_2$.

Since z was arbitrary in $\Omega_1 \cap \Omega_2$, there holds

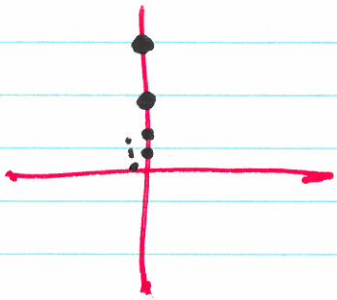
$\mathrm{Int}(\Omega_1 \cap \Omega_2) = \Omega_1 \cap \Omega_2 \Rightarrow \Omega_1 \cap \Omega_2$ is open. $\square$

* An open connected subset of $\mathbb{C}$ is called a domain.

* A set whose interior is a domain is called a region (B-C terminology).

* A point $z \in \mathbb{C}$ is called an accumulation point of $\Omega \subseteq \mathbb{C}$ of a set $\Omega \subseteq \mathbb{C}$ if every deleted nbhd of z intersects Ω.

E.g. (1) $\Omega = \left\{ i/2^n \right\}_{n \in \mathbb{N}}$:



0 is the only accumulation pt of Ω: note $0 \notin \Omega$.

E.g. (2) $\Omega = B$, $\bar{B}$ = set of accumulation points.

§15–16 Limits

Let f be a $\mathbb{C}$-valued fn. defined on a deleted nbhd of some $z_0 \in \mathbb{C}$.

$\lim_{z \to z_0} f(z) = w_0$, i.e., "the limit as z approaches z_0 of $f(z)$ is w_0" says:

Given $\varepsilon > 0$ $\exists \delta > 0$

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon.$$

Note that f does not have to be defined at z_0 for this to make sense/ for the limit to exist.

E.g. $\lim_{z \to 0} \frac{\sin z}{z} = 1$.

Even if $f(z_0)$ exist, there may hold $f(z_0) \neq w_0$.

E.g. $f(z) = \begin{cases} 0 & z \neq 0 \\ 1337 & z = 0 \end{cases}$.

RMK: if a limit exists, it is unique.

PF: praco, week 6.