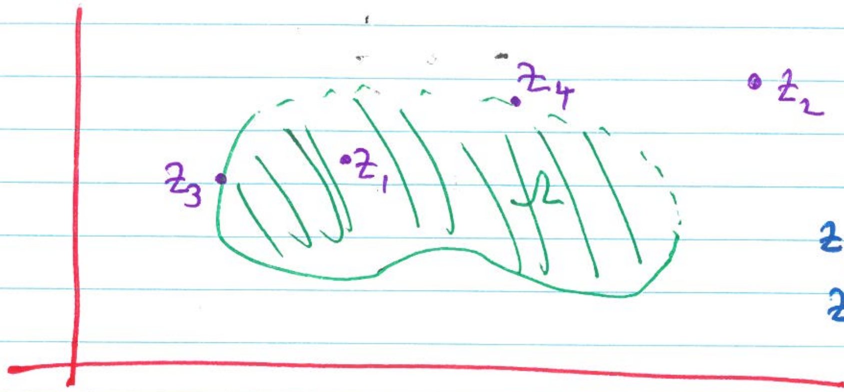
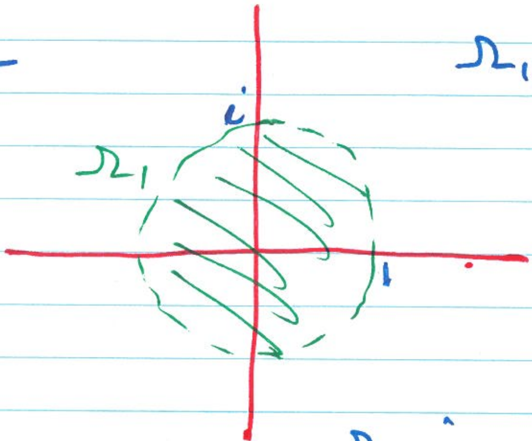
LECTURE 12



z_1 interior pt
z_2 exterior pt
$z_3 \in \partial\Omega$, $z_3 \in \Omega$.
$z_4 \in \partial\Omega$, $z_4 \notin \Omega$.

Ex 1

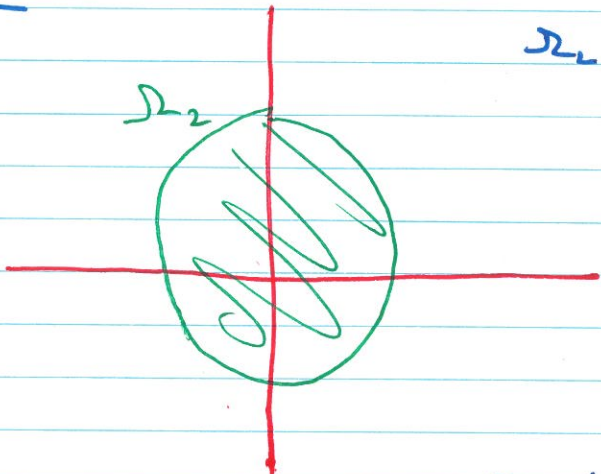


$\Omega_1 = B_1(0) = B_1 = B = \{z : |z| < 1\}$

* $\operatorname{Int} \Omega_1 = \Omega_1$
* $\operatorname{Ext} \Omega_1 = \{z : |z| > 1\}$.
* $\partial\Omega_1 = \{z : |z| = 1\} = S^1$
* $\Omega_1^c = \{z : |z| \geq 1\}$.

Ω_1 is open.

Ex 2

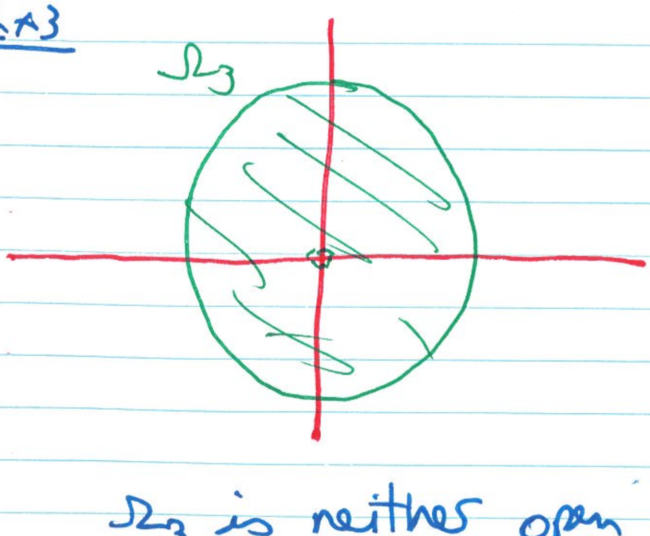


$\Omega_2 = \overline{B}_1(0) = \overline{B}_1 = \overline{B} = \{z : |z| \leq 1\}$.

* $\operatorname{Int} \Omega_2 = \Omega_1$
* $\operatorname{Ext} \Omega_2 = \operatorname{Ext} \Omega_1$.
* $\partial\Omega_2 = S^1 = \partial\Omega_1$.
* $\Omega_2^c = \operatorname{Ext} \Omega_1$.

Ω_2 is closed.

Ex 3



$\Omega_3 = \{z : 0 < |z| \leq 1\}$.

* $\operatorname{Int} \Omega_3 = \{z : 0 < |z| < 1\}$.
* $\operatorname{Ext} \Omega_3 = \operatorname{Ext} \Omega_1$
* $\partial\Omega_3 = S^1 \cup \{0\}$
* $\Omega_3^c = \operatorname{Ext} \Omega_1 \cup \{0\}$.

Ω_3 is neither open nor closed.

<u>Note</u>: Ω_1 is open, Ω_1^c is closed.

Ω_2 is closed, Ω_2^c is open.

Ω_3 & Ω_3^c are neither open nor closed.

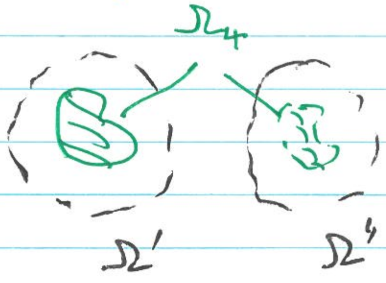
Only clopen sets in $\mathbb{C}$ are $\emptyset$ & $\mathbb{C}$.

closed & open.

* $\Omega \subseteq \mathbb{C}$ is <u>connected</u> if there do not exist non-empty, disjoint open sets Ω' & Ω'' s.t.

$\Omega \subseteq \Omega' \cup \Omega''$ &

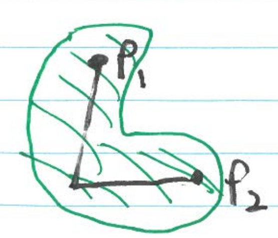
$\Omega' \cap \Omega \neq \emptyset$ & $\Omega'' \cap \Omega \neq \emptyset$.



Ω_4 is not connected i.e., it is disconnected.

Ω_5 is connected, as are Ω_1, Ω_2 & Ω_3.

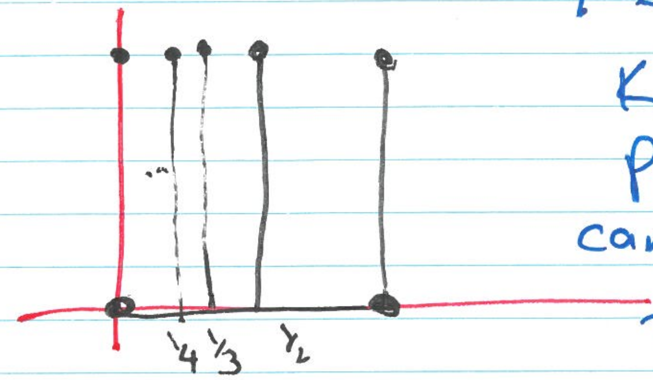
* $\Omega \subseteq \mathbb{C}$ is piecewise affinely path connected if any two points in Ω can be connected by a finite # of line segments <u>in Ω</u>, joined end-to-end.



For open sets in $\mathbb{C}$, the two definitions are <u>equivalent</u>.

In general, the 2 definitions are not equivalent.

Consider $K = \bigcup_{n=1}^{\infty} \{\frac{1}{n} + yi \mid 0 \leq y \leq 1\} \cup \{x + 0i \mid 0 \leq x \leq 1\} \cup \{i\}$



K is connected, but not p.w.a.p.c. as you can't get a path from i to any other point in K.

Example: If Ω_1 & Ω_2 are open in $\mathbb{C}$, then so is $\Omega_1 \cap \Omega_2$.

Pf: If $\Omega_1 \cap \Omega_2 = \emptyset$, we are done ($\emptyset$ is open).

Otherwise, for $z \in \Omega_1 \cap \Omega_2$:

$z \in \Omega_1 \Rightarrow \exists \varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(z) \subset \Omega_1$ ✱

$z \in \Omega_2 \Rightarrow \exists \varepsilon_2 > 0$ s.t. $B_{\varepsilon_2}(z) \subset \Omega_2$ ☺,

since Ω_1, Ω_2 are open.

So, set $\varepsilon = \underbrace{\min}_{\text{minimum}}\{\varepsilon_1, \varepsilon_2\}$: note $\varepsilon > 0$

$B_\varepsilon(z) \subset \Omega_1$ by ✱ & $B_\varepsilon(z) \subset \Omega_2$ by ☺.

So $B_\varepsilon(z) \subset \Omega_1 \cap \Omega_2$.

Since z was arbitrary in $\Omega_1 \cap \Omega_2$, there holds $\text{Int}(\Omega_1 \cap \Omega_2) = \Omega_1 \cap \Omega_2 \Rightarrow \Omega_1 \cap \Omega_2$ is open. $\square$