LECTURE 13

§17 (8Ed §16) Limit Theorems

Suppose $f(\underset{x+iy}{\underset{\shortparallel}{z}}) = u(x,y) + iv(x,y)$

Put $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$

Then

$$\lim_{z \to z_0} f(z) = w_0 \iff \begin{cases} \displaystyle\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \& \\ \displaystyle\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \end{cases}$$

Thm 2 Suppose $\displaystyle\lim_{z\to z_0} f(z) = w_0$ & $\displaystyle\lim_{z\to z_0} g(z) = \xi_0$ & $\lambda \in \mathbb{C}$. Then:

① $\displaystyle\lim_{z\to z_0} (f \pm g)(z) = w_0 \pm \xi_0$;

② $\displaystyle\lim_{z\to z_0} (\lambda f)(z) = \lambda w_0$;

③ $\displaystyle\lim_{z\to z_0} (fg)(z) = w_0 \xi_0$;

④ $\displaystyle\lim_{z\to z_0} \left(\frac{f}{g}\right)(z) = w_0/\xi_0$ as long as $\xi_0 \neq 0$

Limits involving ∞

Recall in $\mathbb{R}$:

$\lim_{x \to x_0} f(x) = \infty$: Given $M > 0$ $\exists\, \delta > 0$ s.t. $0 < |x - x_0| < \delta \Rightarrow f(x) > M$.

Analogously for $\lim_{x \to x_0} f(x) = -\infty$. ☆

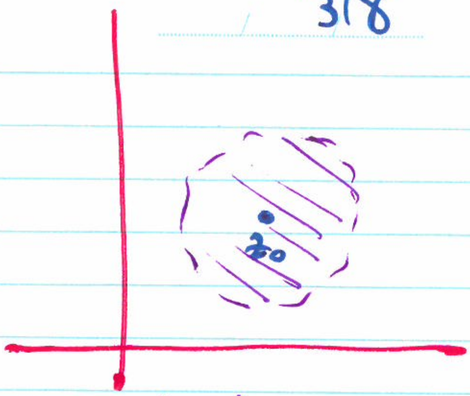
Similarly $\lim_{x \to \infty} f(x) = \lambda \in \mathbb{R}$ says:

Given $\varepsilon > 0$ $\exists M$ s.t. $x > M \Rightarrow |f(x) - \lambda| < \varepsilon$.

Analogously define $\lim_{x \to -\infty} f(x) = \lambda$ ☆

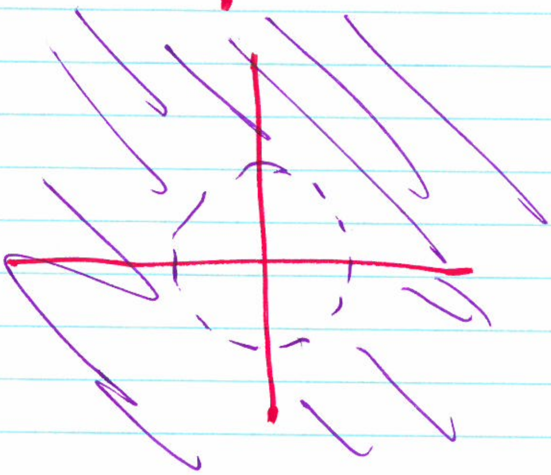
Combine for $\lim_{x \to \pm\infty} f(x) = \pm\infty$, ☆

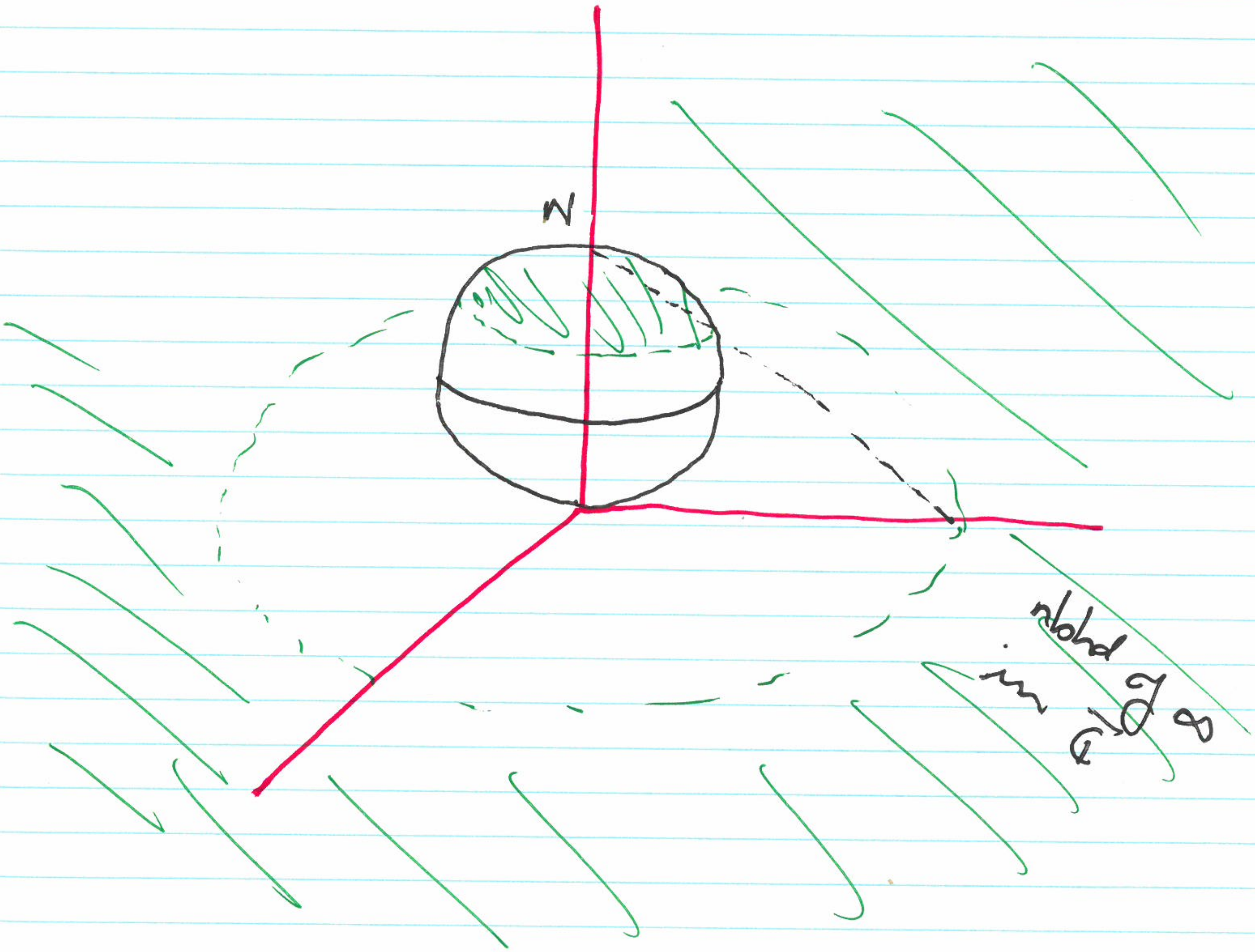
In $\mathbb{C}$: a nbhd of $z_0 \in \mathbb{C}$ is



This is also a nbhd of $z_0 \in \overline{\mathbb{C}}$, $z_0 \neq \infty$.

A nbhd of ∞ in $\overline{\mathbb{C}}$ has the form $\{z : |z| > M\}$.





"close to ∞" $\Leftrightarrow$ $|z|$ is large $\Leftrightarrow$ $1/|z|$ is small.

Keeping this in mind:

* $\lim\limits_{z \to z_0} f(z) = \infty$ means $\lim\limits_{z \to z_0} 1/f(z) = 0$;

* $\lim\limits_{z \to \infty} f(z) = w_0 \in \mathbb{C}$ means $\lim\limits_{z \to 0} f(1/z) = w_0$;

* $\lim\limits_{z \to \infty} f(z) = \infty$ means $\lim\limits_{z \to 0} 1/f(1/z) = 0$.

Ex 1 Show $\lim\limits_{z \to -1} \dfrac{iz+3}{z+1} = \infty$. $f(z)$.

Note f defined on $\mathbb{C} \setminus \{-1\}$.

To show the limit WTS $\lim\limits_{z \to -1} 1/f(z) = 0$ (*)

$$1/f(z) = 1\Big/\frac{iz+3}{z+1} = \frac{z+1}{iz+3}.$$

LHS of (*) $= \lim\limits_{z \to -1} 1/f(z) = \lim\limits_{z \to -1} \dfrac{z+1}{iz+3}$

$$= \frac{\lim\limits_{z \to -1} z+1}{\lim\limits_{z \to -1} iz+3}$$

$$= \frac{0}{3-i} = 0$$

$=$ RHS of (*)

§19 ff (8 Ed §18 ff) Continuity.

f $\mathbb{C}$-valued, defined in a nbhd of $z_0 \in \mathbb{C}$.
f is continuous at z_0 $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$, if
given $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

Basic results:

(1) If $f : \Omega \to U$ & $g : U \to W$ are cts, so is $g \circ f : \Omega \to W$ (g composed with f, $(g \circ f)(z) = g(f(z))$).

(2) If f is cts & nonzero at z_0, then $\exists\, \varepsilon > 0$ s.t. $f(z) \neq 0$ on $B_\varepsilon(z_0)$. Pf: in pracs.

(3) $f : x+iy \mapsto u(x,y) + iv(x,y)$ is cts $\Leftrightarrow$ u & v are cts.

(4) Obvious analogue of Thm 2 from §17 hold.

limit in $\mathbb{C}$.

Differentiability in $\mathbb{C}$

$f: \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$, consider

$$\lim_{\zeta \to 0} \frac{f(z_0 + \zeta) - f(z_0)}{\zeta} \quad (1)$$

If this limit exists, it defines $f'(z_0)$.

To fix ideas, consider $z_0 \in \operatorname{Int} \Omega$.

cf. situation in $\mathbb{R}$:

$f: \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}$, $\displaystyle\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$, if it exists, defines $f'(x_0)$.

Write Δz for ζ in (1):

$$(1) \Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (2)$$

Write $w = f(z)$

Then $\Delta w = f(z_0 + \Delta z) - f(z_0)$

So (2) $\Rightarrow$ $\displaystyle f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}(z_0)$ (3)

Note (1) – (3) are equivalent.

§21 Cauchy-Riemann

$f: \underset{x+iy}{\overset{z}{\Downarrow}} \mapsto w = u(x,y) + i v(x,y).$

Suppose f is differentiable at $z_0 = x_0 + iy_0$.
Set $\Delta z = \Delta x + i\Delta y$ (1) & recall

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \quad (1)'$$

Key point: derivative is independent of how $\Delta z \to 0$ (K)

Note: $\Delta w = f(z_0 + \Delta z) - f(z)$

$$= u(x_0+\Delta x, y_0+\Delta y) + i v(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - i v(x_0,y_0) \quad (2)$$

Note, (1)' $\Rightarrow$

$$f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \operatorname{Re}\frac{\Delta w}{\Delta z} + i \lim_{(\Delta x,\Delta y)\to(0,0)} \operatorname{Im}\frac{\Delta w}{\Delta z} \quad (3)$$

As per (K), value of the limit is independent of how $(\Delta x, \Delta y) \to (0,0)$.

To start, let $(\Delta x, \Delta y) \to (0,0)$ along the x-axis, i.e., consider Δz o.t.f. $(\Delta x, 0)$, $\Delta x \neq 0$.

So ② $\Rightarrow$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$+ i \; \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}.$$

Hence

$$\lim_{\substack{(\Delta x, \Delta y)\to(0,0) \\ \text{on } x\text{-axis}}} \operatorname{Re} \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0) \quad (4)$$

$$\& \lim_{\substack{(\Delta x, \Delta y)\to(0,0) \\ \text{on } x\text{-axis}}} \operatorname{Im} \frac{\Delta w}{\Delta z} = v_x(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0) \quad (5)$$