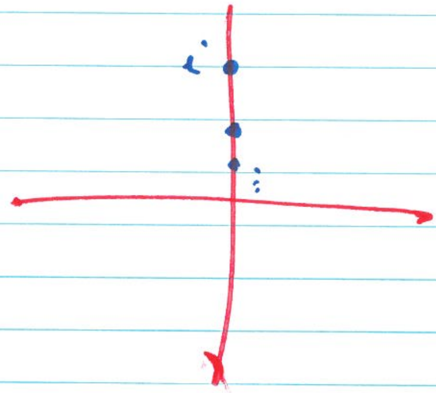
LECTURE 13

* An open, connected subset of $\mathbb{C}$ is called a domain.

A set whose interior is a domain is called a region (B-C terminology).

* A pt $z \in \mathbb{C}$ is called an accumulation point of a set $\Omega \subseteq \mathbb{C}$ is every deleted nbhd of z intersects Ω.

E.g. 1 $\Omega = \{ i/2^n \}_{n \in \mathbb{N}_0}$:



0 is the only accumulation point of Ω.

2 $\Omega = B_1$: Set of accumulation pts $= \overline{B_1}$.

§15–§16 Limits

Let f be a $\mathbb{C}$-valued f^n, defined on a deleted nbhd of some $z_0 \in \mathbb{C}$.

$\lim_{z \to z_0} f(z) = w_0$, i.e., "the limit as z approaches z_0 of $f(z) = w_0$", says:

Given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon.$$

Note: f does not have to be defined at z_0:

E.g., $\lim_{z \to 0} \frac{\sin z}{z} = 1$.

Note e.g. for $f(z) = \begin{cases} 0 & z \neq 0 \\ 1337 & z = 0 \end{cases}$,

$\lim_{z \to 0} f(z) = 0$.

RMK: if a limit exists, it is unique.

Pf: Applied sessions, week 5.

§17 (§8d §16) Limit Th^ms

Suppose $\underset{\overset{\shortparallel}{x+iy}}{f(z)} = u(x,y) + iv(x,y)$.

Put $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$

Then

$$\lim_{z \to z_0} f(z) = w_0 \iff \begin{cases} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \\ \& \\ \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0. \end{cases}$$

Thm. 2 Suppose $\lim_{z\to z_0} f(z) = w_0$ & $\lim_{z\to z_0} g(z) = \xi_0$, & $\lambda \in \mathbb{C}$. Then:

1. $\lim_{z\to z_0} (f \pm g)(z) = w_0 \pm \xi_0$;
2. $\lim_{z\to z_0} (\lambda f)(z) = \lambda w_0$;
3. $\lim_{z\to z_0} (fg)(z) = w_0 \xi_0$;
4. $\lim_{z\to z_0} \left(\frac{f}{g}\right)(z) = w_0/\xi_0$. as long as $\xi_0 \neq 0$.

Limits involving ∞.

Recall in $\mathbb{R}$:

$\lim_{x\to x_0} f(x) = \infty$ means: Given $M>0$ $\exists\, \delta>0$ s.t. $0<|x-x_0|<\delta \Rightarrow f(x) > M$.

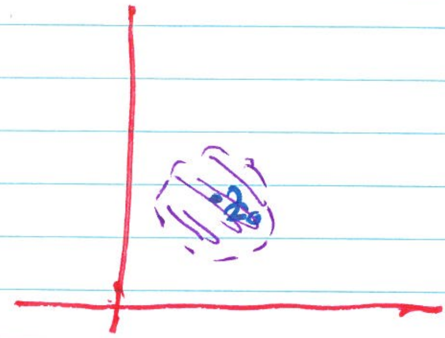
Analogously for $\lim_{x\to x_0} f(x) = -\infty$. ☆

Similarly, $\lim_{x\to\infty} f(x) = \lambda \in \mathbb{R}$ says:

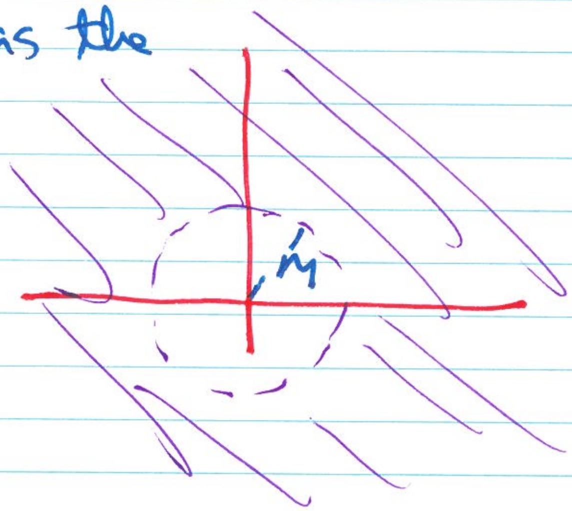
Given $\varepsilon>0$ $\exists\, M>0$ s.t. $x>M \Rightarrow |f(x)-\lambda|<\varepsilon$.

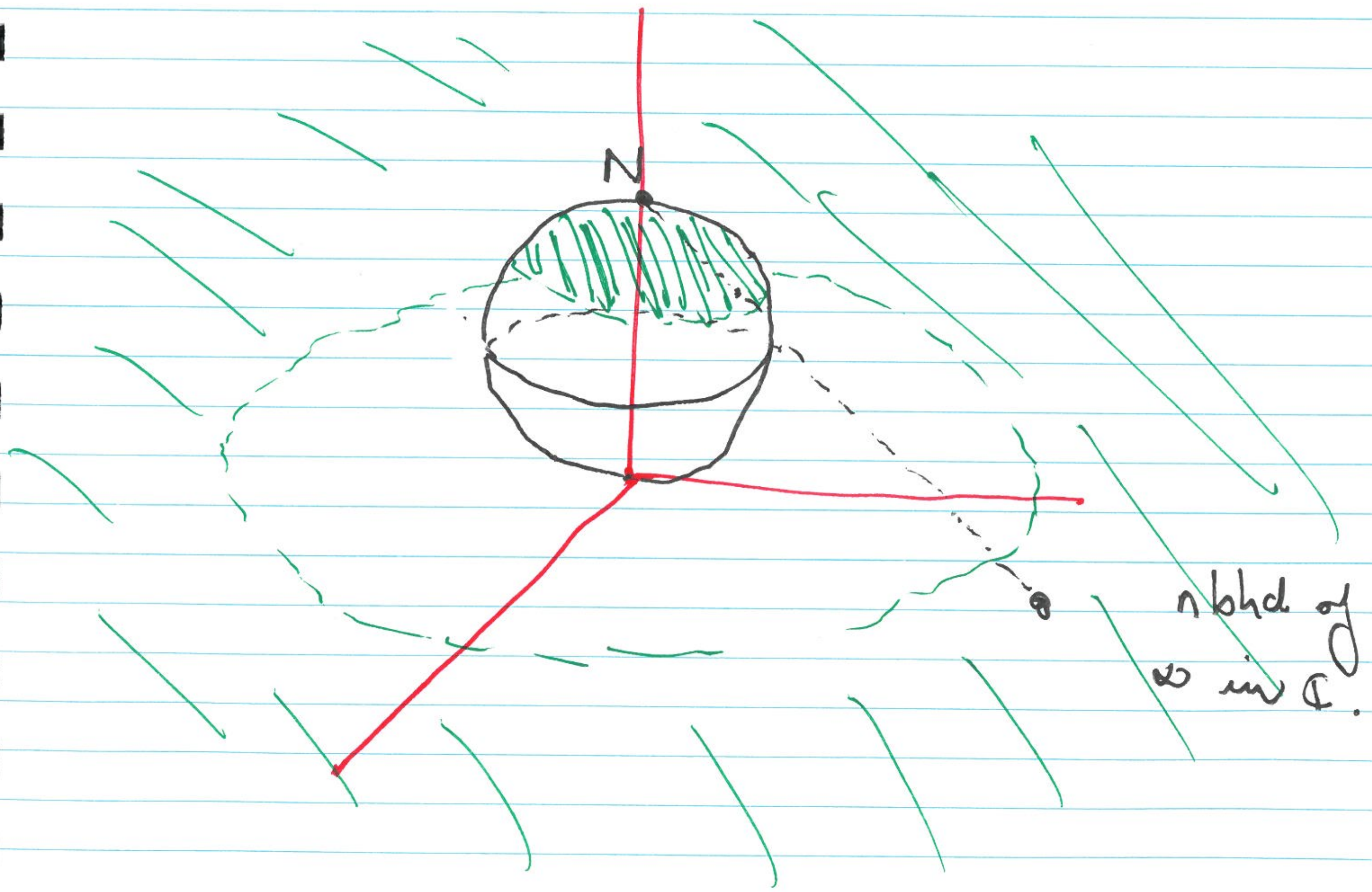
Define $\lim_{x\to -\infty} f(x) = \lambda$, $\lim_{x\to\infty} f(x) = \infty$ etc. analogously. ☆

In $\mathbb{C}$, a nbhd of $z_0 \in \mathbb{C}$ is



A nbhd of ∞ in $\hat{\mathbb{C}}$ has the form $\{z : |z| > M\}$





"close to ∞" $\Leftrightarrow$ $|z|$ is large $\Leftrightarrow$ $1/|z|$ is small.

Keeping that in mind, limits in $\mathbb{C}$ are defined as follows:

* $\lim_{z \to z_0} f(z) = \infty$ means $\lim_{z \to z_0} 1/f(z) = 0$;

* $\lim_{z \to \infty} f(z) = w_0$ means $\lim_{z \to 0} f(1/z) = w_0$;

* $\lim_{z \to \infty} f(z) = \infty$ means $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$.

E.g: 1 Show $\lim_{z \to -1} \frac{iz+3}{z+1} = \infty$. $f(z)$.

Ans: f is defined on $\mathbb{C} \setminus \{-1\}$.

WTS $\lim_{z \to -1} \frac{1}{f(z)} = 0$ (*)

$$\frac{1}{f(z)} = 1 \Big/ \frac{iz+3}{z+1} = \frac{z+1}{iz+3}.$$

$$\text{LHS of (*)} = \lim_{z \to -1} \frac{z+1}{iz+3}$$

$$= \frac{\lim_{z \to -1} (z+1)}{\lim_{z \to -1} (iz+3)}$$ assuming these limits exist.

$$= 0/(3-i) = \text{RHS of (*)}.$$

$g(z)$.

Ex 2 find $\lim_{z\to\infty} \dfrac{2z^3-1}{z^2-1}$

Claim: $\lim_{z\to\infty} g(z) = \infty$.

So, WTS: $\lim_{z\to 0} 1/g(1/z) = 0$.

$$1/g(1/z) = 1\Big/\frac{2(1/z^3)-1}{1/z^2-1}$$

$$= \frac{1/z^2-1}{2(1/z^3)-1}\cdot\frac{z^3}{z^3}$$

$$= \frac{z-z^3}{2-z^3} \to 0/2 \text{ as } z\to 0 \text{ as req}^d.$$

§19 ff (8 Ed §18 ff.) Continuity:

f $\mathbb{C}$-valued, defined in a nbhd of $z_0\in\mathbb{C}$.

f is continuous at z_0, $\boxed{\lim_{z\to z_0} f(z) = f(z_0)}$,

if given $\varepsilon>0$ $\exists\ \delta>0$ s.t.

$$|z-z_0|<\delta \Rightarrow |f(z)-f(z_0)|<\varepsilon.$$

Basic results:

1. If $f: \Omega\to U$ & $g: U\to W$ are cts, so is $g\circ f: \Omega\to W$ (g composed with f, $(g\circ f)(z) = g(f(z))$).
2. If f is cts & nonzero at z_0, then $\exists\ \varepsilon>0$ s.t. $f(z)\neq 0$ on $B_\varepsilon(z_0)$. Pf: Applied sessions, wk. 5.
3. $f: x+iy \mapsto u(x,y)+iv(x,y)$ is cts $\Leftrightarrow$ u & v are cts
4. Obvious analogues of Th^{m}s 1 & 2 from §17 hold.