LECTURE 14

Differentiability in $\mathbb{C}$

limit in $\mathbb{C}$

$f : \Omega \Rightarrow \mathbb{C}$ consider
in
$\mathbb{C}$

$$\lim_{\xi \to 0} \frac{f(z_0+\xi) - f(z_0)}{\xi} \qquad (1)$$

If this limit exists, it defines $f'(z_0)$. To fix ideas, consider $z_0 \in \operatorname{Int} \Omega$.

cf. situation in $\mathbb{R}$:

$f : \Omega \to \mathbb{R}$: $\lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$, if it exist, defines $f'(x_0)$).
in
$\mathbb{R}$

Write Δz for ξ in (1):

$$(1) \Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \qquad (2)$$

Write $w = f(z)$.

Then $\Delta w = f(z_0+\Delta z) - f(z_0)$

So (2) $\Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z} = \dfrac{dw}{dz}(z_0)$ (3)

Note (1) – (3) are equivalent.

Ex ① $f(z) = 4z^2$

Find the derivative of f from first principles, i.e., using the definition of the derivative.

Put $w = f(z)$, take $z_0 \in \mathbb{C}$.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{4(z_0 + \Delta z)^2 - 4z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{4z_0^2} + 8z_0\cancel{\Delta z} + 4(\Delta z)^{\cancel{2}} - \cancel{4z_0^2}}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} 8z_0 + 4\Delta z$$

$$= 8z_0$$

$$\Rightarrow \quad f'(z) = 8z \text{ on } \mathbb{C}.$$

Ex ② $f(z) = |z|^2$ f' does not exist, except at 0.

See BC §23 Ex 2 (8 Ed §22 Ex 2): we will do soon.

Note: differentiability $\Rightarrow$ continuity
$\not\Leftarrow$

Example for $\not\Leftarrow$: $|z|^2$ (or $|z|$, or others).

Proof $\Rightarrow$: 1071, 2400/2401.

§21 : Cauchy-Riemann

$f : z \mapsto w = u(x,y) + i\,v(x,y)$

$z = x+iy$.

Suppose f is differentiable at $z_0 = x_0 + iy_0$.

Set $\Delta z = \Delta x + i\Delta y$ (1), & recall

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \quad (1)'$$

Key point: derivative is independent of how $\Delta z \to 0$ (K)

Note: $\Delta w = f(z_0 + \Delta z) - f(z_0)$

$$= u(x_0+\Delta x, y_0+\Delta y) + i\,v(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - i\,v(x_0,y_0) \quad (2)$$

Note (1)′ $\Rightarrow$

$$f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \operatorname{Re} \frac{\Delta w}{\Delta z} + i \lim_{(\Delta x,\Delta y)\to(0,0)} \operatorname{Im} \frac{\Delta w}{\Delta z} \quad (3)$$

As per (K), value of the limit is independent of how $(\Delta x, \Delta y) \to (0,0)$.

To start, let $(\Delta x, \Delta y) \to (0,0)$ along x-axis, i.e. consider Δz o.t.f. $(\Delta x, 0)$, $\Delta x \neq 0$.

So (2) $\Rightarrow$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\,\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

Hence:

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on the } x \text{ axis}}} \operatorname{Re}\frac{\Delta w}{\Delta z} = u_x(x_0,y_0) = \frac{\partial u}{\partial x}(x_0,y_0) \quad (4)$$

$$\& \lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on the } x \text{ axis}}} \operatorname{Im}\frac{\Delta w}{\Delta z} = v_x(x_0,y_0) = \frac{\partial v}{\partial x}(x_0,y_0) \quad (5)$$

Repeat this for Δz o.t.f. $\Delta z = 0 + i\Delta y$, $\Delta y \neq 0$, i.e., $\Delta z \to 0$ on the y axis.

For such Δz: $\displaystyle \frac{\Delta w}{\Delta z} = -i\left[\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\cancel{i}\Delta y}\right] + \cancel{i}\,\frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{\cancel{i}\Delta y}.$

Hence:

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on } y \text{ axis}}} \operatorname{Re}\frac{\Delta w}{\Delta z} = v_y(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad (6)$$

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on } y \text{ axis}}} \operatorname{Im}\frac{\Delta w}{\Delta z} = -u_y(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0) \quad (7)$$

Keeping in mind (K), {(4) & (6)} & {(5) & (7)} we have:

the Cauchy-Riemann equations:

If $f = u + iv$ is diff^{ble}. at $z_0 = x_0 + iy_0$,

then $\boxed{\begin{array}{l} u_x = v_y \ \& \\ -v_x = u_y \quad \text{at } (x_0, y_0) \end{array}}$ C/R.

Note: we have shown that C/R are necessary for $\mathbb{C}$ differentiability. They are NOT sufficient (later).