LECTURE 14

Repeat the procedure of Lec 13 for Δz o.t.f. $\Delta z = 0 + i\Delta y$, $\Delta y \neq 0$, i.e., $\Delta z \to 0$ on the y-axis.

For such Δz: $$\frac{\Delta w}{\Delta z} = -i\left[\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\cancel{i}\Delta y}\right. \left. + \cancel{i}\,\frac{v(x_0,y_0+\Delta y) - v(x_0,y_0)}{\cancel{i}\Delta y}\right].$$

Hence:

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on } y \text{ axis}}} \operatorname{Re}\frac{\Delta w}{\Delta z} = v_y(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad (6)$$

$$\& \lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on } y \text{ axis}}} \operatorname{Im}\frac{\Delta w}{\Delta z} = -u_y(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0) \quad (7)$$

Keeping in mind (K), {(4) & (6)} & {(5) & (7)}, we have the Cauchy Riemann equations:

If $f = u + iv$ is diff^{ble} at $z_0 = x_0 + iy_0$, then

$$\boxed{\begin{cases} u_x = v_y \ \& \\ -v_x = u_y \end{cases} \text{at } (x_0,y_0).} \quad C/R.$$

Important: we have shown that C/R are necessary for $\mathbb{C}$-differentiability. They are not actually sufficient.

Sufficient cond's for $f'(z_0)$ to exist:

Suppose

* $f = u + iv$ is defined in a nbhd of $z_0 = x_0 + iy_0$;
* u_x, u_y, v_x, v_y are defined & cts in a nbhd of z_0
* C/R hold at (x_0, y_0)

Then $f'(z_0)$ exists.

RMK: $*1, *2 \Rightarrow f$ is cts in a nbhd of z_0.

C/R in polar coords

$z = x + iy = re^{i\theta}$

$x = r\cos\theta$, $y = r\sin\theta$: suppose $f = u + iv$ is $\mathbb{C}$-diff^ble.

Chain rule in 2-D:

$$u_r = u_x \cos\theta + u_y \sin\theta \quad (1)$$
$$u_\theta = -u_x r \sin\theta + u_y r \cos\theta \quad (2)$$
$$v_r = v_x \cos\theta + v_y \sin\theta \quad (3)$$
$$v_\theta = -v_x r \sin\theta + v_y r \cos\theta \quad (4)$$

Combine with C/R in rectangular coords $\Rightarrow$ C/R in polar coords

$$\left.\begin{array}{l} r u_r = v_\theta \\ u_\theta = -r v_r \end{array}\right\}$$

Useful to note: in deriving C/R (rectangular), we noted: $\boxed{f' = u_x + i v_x}$.

Analogously, there holds ☆ $\boxed{f' = e^{-i\theta}(u_r + i v_r)}$

Backtrack: Another example involving limits at ∞:

Calculate $\displaystyle\lim_{z\to\infty} \frac{2z^3-1}{z^2-1}$. $g(z)$

Claim $\displaystyle\lim_{z\to\infty} g(z) = \infty$. WTS: $\displaystyle\lim_{z\to 0} 1/g(1/z) = 0$.

$$1/g(1/z) = 1\Big/\frac{2(1/z)^3-1}{(1/z)^2-1}$$

$$= \frac{(1/z)^2-1}{2(1/z)^3-1} \cdot \frac{z^3}{z^3}$$

$$= \frac{z-z^3}{2-z^3} \to 0/2 = 0 \text{ as } z\to 0$$

as req^d.

Further on derivatives.

* Some calculations/properties are completely analogous to those in $\mathbb{R}$!

E.g., Find the derivative of f from first principles, i.e., using the definition of the derivative, for $f(z) = 4z^2$.

Put $w = f(z)$, take $z_0 \in \mathbb{C}$.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{4(z_0 + \Delta z)^2 - 4z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{4z_0^2} + 8z_0\cancel{\Delta z} + (\Delta z)^{\cancel{2}} - \cancel{4z_0^2}}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} 8z_0 + 4\Delta z$$

$$= 8z_0$$

$\Rightarrow f'(z) = 8z$ on $\mathbb{C}$.

* Some aren't.

For $g(z) = |z|^2$, Claim: $g'(z)$ does not exist, except at 0.

Pf: (see also §23 Ex 2 (8Ed §22 Ex 2)).

$$g(z) = \underbrace{(x^2 + y^2)}_{u} + \underbrace{0}_{v} i$$

C/R_{I}: $u_x = v_y \Rightarrow 2x = 0 \Rightarrow x = 0$

C/R_{II}: $u_y = -v_x \Rightarrow 2y = 0 \Rightarrow y = 0$.

So g is only possibly diff^{ble} at $(0,0)$

Check: suff condⁿs for $\mathbb{C}$-diff$^{\text{bility}}$ satisfied at $(0,0)$

Hence g is differentiable precisely at $0+0i$.

Note: differentiability $\Rightarrow$ continuity.

$$\not\Leftarrow$$

Example for $\not\Leftarrow$: $|z|^2$ (or $|z|$, or others).

Pf of $\Rightarrow$ in MATH1071, 2400/2401.