# LECTURE 14

## Differentiability in $\mathbb{C}$

$f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$ consider

$$\lim_{\xi \to 0} \frac{f(z_0+\xi) - f(z_0)}{\xi} \qquad (1)$$

limit in $\mathbb{C}$.

If this limit exists, it defines $f'(z_0)$. To fix ideas, consider $z_0 \in \operatorname{Int} \Omega$.

Cf. situation in $\mathbb{R}$:

$f : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}$ : $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$, if it exists, defines $f'(x_0)$.

Write $\Delta z$ for $\xi$ in (1):

$$(1) \Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \qquad (2)$$

Write $w = f(z)$.

Then $\Delta w = f(z_0+\Delta z) - f(z_0)$

So (2) $\Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}(z_0)$ (3)

Note: (1) – (3) are equivalent.

## §21 Cauchy-Riemann

$f: z \longmapsto w = u(x,y) + iv(x,y)$
$z = x+iy.$

Suppose $f$ is differentiable at $z_0 = x_0 + iy_0$.
Set $\Delta z = \Delta x + i\Delta y$ ① , & recall

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \quad ①'.$$

Key point: derivative is independent of how $\Delta z \to 0$. Ⓚ

Note: $\Delta w = f(z_0 + \Delta z) - f(z)$

$$= u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - iv(x_0,y_0) \quad ②$$

Note ①' $\Rightarrow$

$$f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \mathrm{Re}\frac{\Delta w}{\Delta z} + i \lim_{(\Delta x,\Delta y)\to(0,0)} \mathrm{Im}\frac{\Delta w}{\Delta z} \quad ③$$

As per Ⓚ, the value of the limit is independent of how $(\Delta x, \Delta y) \to (0,0)$.

To start, let $(\Delta x, \Delta y) \to (0,0)$ along the $x$-axis, i.e., consider $\Delta z$ o.t.f. $(\Delta x, 0)$, $\Delta x \neq 0$.

So (2) $\Rightarrow$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\,\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}.$$

Hence:

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{along the } x \text{ axis}}} \operatorname{Re}\frac{\Delta w}{\Delta z} = u_x(x_0,y_0) = \frac{\partial u}{\partial x}(x_0,y_0) \quad (4)$$

$$\& \lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{along } x \text{ axis}}} \operatorname{Im}\frac{\Delta w}{\Delta z} = v_x(x_0,y_0) = \frac{\partial v}{\partial x}(x_0,y_0) \quad (5)$$

Repeat this for $z$ o.t.f. $\Delta z = 0 + i\Delta y$, $\Delta y \neq 0$, i.e. $\Delta z \to 0$ on the $y$ axis.

For such $\Delta z$:

$$\frac{\Delta w}{\Delta z} = -i\left[\frac{u(x_0,y_0+\Delta y) - u(x_0,y_0)}{\cancel{i}\Delta y}\right] \cancel{+i}\,\frac{v(x_0,y_0+\Delta y) - v(x_0,y_0)}{\cancel{i}\Delta y}$$

Hence:

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on } y \text{ axis}}} \operatorname{Re}\frac{\Delta w}{\Delta z} = v_y(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad (6)$$

$$\& \lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \text{on } y \text{ axis.}}} \operatorname{Im}\frac{\Delta w}{\Delta z} = -u_y(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0). \quad (7)$$

Keeping in mind (K) & $\{$(4) & (6)$\}$ & $\{$(5) & (7)$\}$, we have the Cauchy Riemann equations:

If $f = u + iv$ is diff$^{ble}$ at $z_0 = x_0 + iy_0$

then $\boxed{\begin{array}{ll} u_x = v_y & \text{at } (x_0, y_0) \\ \& \ u_y = -v_x & \end{array}}$ C/R.

Note: we have shown that C/R are necessary for $\mathbb{C}$-differentiability. They are not sufficient (later).

Ex① $f(z) = 4z^2$

Find the derivative of $f$ from first principles, i.e., using the def$^n$ of the derivative.

Put $w = f(z)$, take $z_0 \in \mathbb{C}$.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{4(z_0 + \Delta z)^2 - 4z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{4z_0^2} + 8z_0 \cancel{\Delta z} + 4(\Delta z_0)^{\cancel{2}} - \cancel{4z_0^2}}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} 8z_0 + 4\Delta z$$

$$= 8z_0.$$

$\Rightarrow f'(z) = 8z$ on $\mathbb{C}$.

Ex (2) $f(z) = |z|^2$.

$f'(z)$ does not exist, except at 0.

Note differentiability $\Rightarrow$ continuity

$\not\Leftarrow$

Example for $\not\Leftarrow$ : $|z|^2$ (or $|z|$, or others).

Pf of $\Rightarrow$ 1071, 2400/2401.