# LECTURE 15

## Formulae (cf. $f:\mathbb{R}\to\mathbb{R}$)

* $\frac{d}{dz}(c) = 0$ $\quad c\in\mathbb{C}$ constant.
* $\frac{d}{dz}(z^n) = nz^{n-1}$ $\quad n\in\mathbb{Z}$.
* $\frac{d}{dz}(e^z) = e^z$
* $\frac{d}{dz}(\sin z) = \cos z$ ; $\frac{d}{dz}(\cos z) = -\sin z$
* other trig, hyperbolic etc.

For $f, g$ diff$^{ble}$:

$$(f\pm g)' = f' \pm g';$$
$$(fg)' = fg' + f'g;$$
$$(f/g)' = \frac{gf' - fg'}{g^2} \quad g\neq 0.$$

Chain rule: $f$ diff$^{ble}$ at $z_0$ & $g$ diff$^{ble}$ at $f(z_0)$, then $g\circ f$ is diff$^{ble}$ at $z_0$ & $(g\circ f)'(z_0) = g'(f(z_0))\cdot f'(z_0)$.

Write $\frac{dg}{dz} = \frac{dg}{dw}\frac{dw}{dz}$ where $w = f(z)$.

E.g. $\frac{d}{dz}(74z^2+9)^{171} = 171(74z^2+9)^{170}\cdot 2\cdot 74z$
$= 25308(74z^2+9)^{170} z$.

<u>Def<sup></sup>:</u> $f: \Omega \xrightarrow{\subseteq \mathbb{C}} \mathbb{C}$ is <u>analytic</u> at $z_0$ if it is differentiable on a neighbourhood of $z_0$.

A $f^n$ is <u>singular</u> at $z_0$ if it is <u>NOT</u> analytic at $z_0$, but <u>is</u> analytic at some pt. in <u>every</u> nbhd of $z_0$.

E.g. $z \mapsto 1/z$ is analytic on $\mathbb{C}_*$, & is singular at $0$.

A $f^n$ is <u>entire</u> if it is analytic on $\mathbb{C}$, e.g. $e^z$, polynomials, sin, cos, sinh, cosh, etc.

Important to note: if a $f^n$ is differentiable at precisely one pt, it is not analytic there or anywhere, e.g. $z \mapsto |z|^2$.

## Further on derivatives

$\frac{d}{dz}(\log z)$ on $\mathbb{C}_*$.

(maybe "$\frac{d}{dz}(\log z)$" would be a better starting point, as $\log$ is multi-valued.

Recall $\log z = \ln|z| + i \arg z$

$$\overset{\shortparallel}{re^{i\theta}} = \ln r + i\theta$$

$u = \ln r$, $v = \theta$

$\Rightarrow$ $u_r = 1/r$, $u_\theta = 0$; $v_r = 0$, $v_\theta = 1$.

$$C/R: \begin{cases} * \; r u_r = v_\theta & \checkmark \\ * \; u_\theta = -r v_r. & \checkmark \end{cases}$$

So, sufficient conditions for complex differentiability are satisfied on any subset of $\mathbb{C}_*$ s.t.

$$\alpha < \theta < \alpha + 2\pi \qquad \alpha \text{ fixed in } \mathbb{R}. ☺$$

On such a set, $\log$ (or, precisely, the single-valued $f^n$ we obtain from $\log$ via ☺) is diff$^{ble}$.

& Lec. 14 $\Rightarrow \frac{d}{dz}(\log z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(\frac{1}{r} + 0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

E.g. $\frac{d}{dz} \operatorname{Log} z = 1/z$ for $-\pi < \operatorname{Arg} z < \pi$, $|z| > 0$.

For $f(z) = z^c$, $c \in \mathbb{C}_*$ fixed : on $\mathbb{C}_*$

$$f(z) = \exp(c \log z) \ \Delta$$

$$f'(z) = \exp(c \log z) \cdot c/z \quad (*)$$

$$= z^c \cdot c/z = c z^{c-1} \quad (**)$$

(**) is valid on any domain o.t.f. $\{z : |z| > 0 \ , \alpha < \arg z < \alpha + 2\pi\}$, due to the need to choose a branch of log in (*).

RMK: try for $g(z) = c^z$ $\Delta$.

# Notation from real analysis

$\Omega \subseteq \mathbb{R}^n$ $\quad n \geq 1$.

* $C(\Omega) = C^0(\Omega) = \{$cts f$^n$s $: \Omega \to \mathbb{R}\}$.

* $C^k(\Omega) = \{$f$^n$s $\Omega \to \mathbb{R}$ s.t. f and all its derivatives / partial derivatives of order $\leq k$ exist & are cts on $\Omega\}$

Note: 0th order derivative of f is f.

* $C^\infty(\Omega) = \{f : \Omega \to \mathbb{R}$ s.t. f & all derivs / partial derivs of all orders exist & are cts on $\Omega.\}$, a.k.a. smooth f$^n$s.

$C^\omega(\Omega)$ = real analytic f$^n$s on $\Omega$:

$f \in C^\omega(\Omega)$ says, at every pt $x_0 \in \Omega$:

(i) f has a power series expansion about $x_0$ (namely, its Taylor series);

(ii) f is given by its power series expansion, i.e., the power series converges to f on some nbhd of $x_0$.

(i) $\Rightarrow$ $f \in C^\infty(\Omega)$

(i) $\not\Rightarrow$ (ii) in $\mathbb{R}^n$.

example: next time.