LECTURE 15

Sufficient conditions for $f'(z_0)$ to exist.

Suppose:

*1 $f = u + iv$ is defined in a nbhd of $z_0 = x_0 + iy_0$;

*2 u_x, u_y, v_x, v_y are defined & cts in a nbhd of z_0;

*3 C/R hold at (x_0, y_0).

Then $f'(z_0)$ exists.

Rmk: *1 + *2 $\Rightarrow f$ is cts in nbhd of z_0.

Formulae (cf. $f:\mathbb{R}\to\mathbb{R}$)

- $\frac{d}{dz}(c) = 0$ $\quad$ c constant in $\mathbb{C}$
- $\frac{d}{dz}(z^n) = nz^{n-1}$ $\quad$ $n\in\mathbb{Z}$
- $\frac{d}{dz}e^z = e^z$
- $\frac{d}{dz}\sin z = \cos z$; $\frac{d}{dz}\cos z = -\sin z$
- other trig, hyperbolic, etc.

For f, g diff^{ble}:

$$(f\pm g)' = f'\pm g' ;$$
$$(fg)' = fg' + f'g ;$$
$$(f/g)' = \frac{gf' - fg'}{g^2} \qquad g\neq 0.$$

Chain rule: f diff^{ble} at z_0 & g diff^{ble} at $f(z_0)$, then $g\circ f$ is diff^{ble} at z_0 &

$$(g\circ f)'(z_0) = g'(f(z_0))\cdot f'(z_0).$$

Write $\frac{dg}{dz} = \frac{dg}{dw}\frac{dw}{dz}$ where $w = f(z)$.

E.g. $\frac{d}{dz}(74z^2+9)^{171} = 171(74z^2+9)^{170}\cdot 2\cdot 74\cdot z$

$= 25308(74z^2+9)^{170}\cdot z$.

C/R in polar coords

$z = x + iy = re^{i\theta}$

$x = r\cos\theta$, $y = r\sin\theta$ $f = u + iv$ is $\mathbb{C}$-diff^{ble}.

Chain rule (in 2D) $\Rightarrow$

$u_r = u_x\cos\theta + u_y \sin\theta$ (1)

$u_\theta = -u_x r\sin\theta + u_y r\cos\theta$ (2)

$v_r = v_x\cos\theta + v_y\sin\theta$ (3)

$v_\theta = -v_x r\sin\theta + v_y r\cos\theta$ (4)

Combine with C/R from Lec. 14 ★

$\Rightarrow$ C/R in polar coords:

$$\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}.$$

Useful to note: recall (K) from Lec 14 $\Rightarrow$

If f' exists, then $\boxed{f' = u_x + i v_x}$ (5)

& similarly

$$\boxed{f'(z) = e^{-i\theta}(u_r + i v_r)}. \quad (6)$$

Another approach to C/R

Formally $(x,y) \to (z,\bar{z})$

$z = x+iy \qquad \bar{z} = x-iy$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x}$$

$$= \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \qquad (7)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y}$$

$$= i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial \bar{z}} \qquad (8)$$

$(7) - i(8)$ $\quad \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 2\frac{\partial f}{\partial z}$

$$\Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$(7) + i(8)$ $\quad \star \Rightarrow \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$

$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ are called the Wirtinger operators.

Ex: $f(z) = z^n = (x+iy)^n \qquad n \in \mathbb{Z}.$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy)^n$$

$$= \frac{1}{2}n(x+iy)^{n-1}(1-i^2)$$

$$= n(x+iy)^{n-1}$$

$$= nz^{n-1} = f'(z).$$

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \star$$

For $f = u + iv$ complex diff^{ble}:

$$\frac{1}{2}\frac{\partial f}{\partial x} = \frac{1}{2}(u_x + iv_x) \overset{C/R_I}{=} \frac{1}{2}(v_y - iu_y)$$

$$= \frac{-i}{2}(u_y + iv_y) = \frac{-i}{2}\frac{\partial f}{\partial y}.$$

$$\Rightarrow \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial \bar{z}} = 0.} \qquad C/R \text{ version } \underline{II}.$$

<u>Application</u> When is $g(z) = |z|^2$ diff^{ble}?

Note: $g(z) = z\bar{z} = x^2 + y^2$.

C/R_{II} $\quad \frac{\partial g}{\partial \bar{z}} = 0 \iff z = 0$.

So, the only point at which g is possibly diff^{ble} is 0.

Note $g = \underbrace{(x^2+y^2)}_{u} + i\underbrace{0}_{v}$, so easy to check suff cond^{n}s are satisfied.

So, g is diff^{ble} precisely at 0.

Sidebar: from 5, for diff^{ble} f,

$$\frac{df}{dz} = u_x + iv_x = \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial f}{\partial z}.$$